SOME RESULTS ON LOCAL COHOMOLOGY OF POLYNOMIAL AND FORMAL POWER SERIES RINGS: THE ONE DIMENSIONAL CASE

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Abstract. In this paper, we prove several results on the finiteness of local cohomology of polynomial and formal power series rings. In particular, we give a partial affirmative answer for a question of L. Núñez-Betancourt in [J. Algebra 399 (2014), 770–781].

1. Introduction

The motivation of this paper is the following conjecture of G. Lyubeznik: If R is a regular ring, then each local cohomology module $H^i_I(R)$ has finitely many associated prime ideals. The Lyubeznik conjecture has affirmative answers in several cases: for regular rings of prime characteristic (cf. [7, 9]); for regular local and affine rings of characteristic zero (cf. [8]); for unramified regular local rings of mixed characteristic (cf. [11, 13]) and for smooth $\mathbb{Z}$-algebras (cf. [2]). The method of the proof of these results is considering the module structure of local cohomology over non-commutative rings, D-modules (resp. F-modules). The finiteness of these module structures (for example, finite length) yields the finiteness of $\mathrm{Ass}_S H^i_I(R)$.

Motivated by the above finiteness results, M. Hochster raised the following related question (cf. [14, Question 1.1]):

Question 1. Let $(R, \mathfrak{m}, k)$ be a local ring and S a flat extension of R with regular closed fiber. Then is
$$\mathrm{Ass}_S H^0_{\mathfrak{m}S}(H^i_I(S)) = V(\mathfrak{m}S) \cap \mathrm{Ass}_S H^i_I(S)$$
finite for every ideal $I \subset S$ and for every integer $i \geq 0$?

Suppose S is a flat extension of R with regular fibers. It is worth to note that if Question 1 has an affirmative answer, then the finiteness conditions of $\mathrm{Ass}_S H^i_I(S)$ and $\mathrm{Ass}_R H^i_I(S)$ are equivalent. In [14], L. Núñez-Betancourt gave a positive answer for Question 1 when S is either $R[x_1, ..., x_n]$ or $R[[x_1, ..., x_n]]$ and $\dim R/(I \cap R) \leq 1$. In that paper, he introduced the notion of Σ-finite D-modules. It should be noted that Σ-finite D-modules maybe not have finite length but they have finitely many associated primes. Núñez-Betancourt asked the following question (cf. [14, Question 5.1]).

Question 2. Let $(R, \mathfrak{m}, k)$ be a local ring and S either $R[x_1, ..., x_n]$ or $R[[x_1, ..., x_n]]$. Then is $H^i_{\mathfrak{m}} H^j_J(S)$ Σ-finite for every ideal $J \subset S$ and $i, j \geq 0$?

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Throughout this paper, let R be a commutative Noetherian ring and S be either $R[X_1,...,X_n]$ or $R[[X_1,...,X_n]]$. In Section 3 we modify the definition of Σ-finite D-modules for rings that not necessarily local rings. We prove that $H^j_J(S)$ is Σ-finite for every ideal $J \subseteq S$ satisfying $\dim R/(J\cap R) = 0$ (cf. Proposition 3.7). Applying this result we give a positive answer for Question 2 when $\dim R/(J\cap R) \leq 1$ (cf. Theorem 3.8). Moreover, a finiteness result of associated primes of local cohomology is given (cf. Corollary 3.9).

In Section 4 we consider the following problem.

Question 3. Suppose that $\dim R = 1$ and S is either $R[X_1,...,X_n]$ or $R[[X_1,...,X_n]]$. Is it true that $H^i_J(S)$ has only finitely many associated primes for all ideals J of S and all $i \geq 0$?

By the work of B. Bhatt et al. [2] Question 3 has a positive answer when $S = \mathbb{Z}[x_1,...,x_n]$. The next interesting case of Lyubeznik's conjecture is seem to be the case $S = R[x_1,...,x_n]$ with R is a Dedekind domain (containing the field of rational numbers). This is a special case of Question 3. In this section we will give a partial affirmative answer of Question 3 in the case R contains a field of positive characteristic (cf. Proposition 4.4). It should be noted that H. Dao and the author showed that local cohomology of Stanley-Reisner rings over a field of positive characteristic have only finitely many associated primes, see [4] for a more general result (see also [6]). Finally, the readers are encouraged to [15, 16] for some results about the finiteness of associated primes of local cohomology of polynomial and power series rings over a normal domain containing a field of zero characteristic.

2. Preliminary

In this section we collect some basic facts on rings of differential operators and D-modules. Let R be a Noetherian ring and $S = R[X_1,...,X_n]$ or $S = R[[X_1,...,X_n]]$.

Rings of differential operators. Let $D(S,R)$ (or D if there is no confusion) be the ring of R-linear differential operators of S. The ring $D(S,R)$ is defined by recursion as follows. The differential operators of order zero are the morphisms induced by multiplying by elements in S. An element $\delta \in \mathrm{Hom}_R(S,S)$ is a differential operator of order less than of equal to $k+1$ if $[\delta, s] := \delta\circ s - s\circ\delta$ is a differential operator of order less than or equal to k for every $s \in S = \mathrm{Hom}_S(S,S)$. Notice that $D(S,R)$ is not a commutative ring, but R is contained in the center of $D(S,R)$. In our cases $S = R[X_1,...,X_n]$ or $S = R[[X_1,...,X_n]]$, it is well known that (see [5, Theorem 16.12.1])

$$D(S,R) = S\big[\frac{1}{t!}\frac{\partial^t}{\partial x_i^t} | t \in \mathbb{N}, 1 \leq i \leq n\big] \subseteq \mathrm{Hom}_R(S,S).$$

Homomorphic. Let R' be another ring with $\phi : R \to R'$ a homomorphism of rings. Let S' be either $R'[x_1,...,x_n]$ or $R'[[x_1,...,x_n]]$, respectively. Then ϕ induces a homomorphism between rings of differential operators $\Phi : D(S,R) \to D(S',R')$. In particular, we have a natural surjection $D(S,R) \to D(S/IS, R/I)$ for every ideal $I \subset R$.

Example 2.1 (of D-modules). (i) It is well known that S is a D-module.

(ii) Let M be an R-module. Then $M[x_1,...,x_n] \cong R[x_1,...,x_n]\otimes_R M$ (resp. $R[[x_1,...,x_n]]\otimes_R M$ and $M[[x_1,...,x_n]]$) are D-modules. In particular for each $\mathfrak{m} \in \mathrm{Max}(R)$ we have $(R/\mathfrak{m})[x_1,...,x_n]$ (resp. $(R/\mathfrak{m})[[x_1,...,x_n]]$) are D-modules of finite length.

(iii) If M is a D-module then its localization and local cohomology of M are D-modules.
(iv) In [10], Lyubeznik defined the subcategory of the category of $D(S,R)$-modules, says $C(S,R)$, is the smallest subcategory of $D(S,R)$-modules that contains S_f for all $f \in S$ and that is closed under taking submodules, quotients and extensions. In particular, the kernel, image and cokernel of a morphism of $D(S,R)$-modules that belongs to $C(S,R)$ are also objects in $C(S,R)$. Notice that $H^{i_k}_{I_k} \cdots H^{i_1}_{I_1}(S)$ is an object in $C(S,R)$. The critical fact for the study of the finiteness of local cohomology is that every module in $C(S,R)$ has finite length as a D-module provided R is a field (see [10, Corollary 6]).

3. Σ-finite D-modules

First, we give the definition of Σ-finite D-modules. Notice that we do not assume R is local as [14]. Let M be a D-module, we denote by $\mathrm{Fin}(M)$ the set of all D-submodules of M that have finite length. Let N be a D-module of finite length. There is a filtration of submodules $0 = N_0 \subset N_1 \subset \cdots \subset N_h = N$ such that N_i/N_{i-1} is a nonzero simple D-module for all $i = 1, ..., h$. The factors, N_i/N_{i-1}, are the same, up to permutation and isomorphism, for every filtration. We denote that set of factors by $\mathcal{C}(N)$.

Definition 3.1. Let M be a D-module such that $\mathrm{Supp}_R(M) \subseteq \mathrm{Max}(R)$. We say that M is Σ-*finite* if

(i) $\bigcup_{N\in \mathrm{Fin}(M)} N = M$,
(ii) $\bigcup_{N\in \mathrm{Fin}(M)} \mathcal{C}(N)$ is finite, and
(iii) For every $N \in \mathrm{Fin}(M)$ and $L \in \mathcal{C}(N)$, $L \in C(S/\mathfrak{m}S, R/\mathfrak{m})$ for some $\mathfrak{m} \in \mathrm{Max}(M)$.

If M is Σ-finite, we denote $\mathcal{C}(M) := \cup_{N\in \mathrm{Fin}(M)}\mathcal{C}(N)$. It is easy to see that if

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence of Σ-finite D-modules, then $\mathcal{C}(M) = \mathcal{C}(M') \cup \mathcal{C}(M'')$.

Remark 3.2. If M is Σ-finite then $\mathrm{Supp}_R(M)$ is a finite subset of $\mathrm{Max}(R)$. If $\mathrm{Supp}_R(M) = \{\mathfrak{m}_1, ..., \mathfrak{m}_r\} \subseteq \mathrm{Max}(R)$, then $M \cong \Gamma_{\mathfrak{m}_1}(M) \oplus \cdots \oplus \Gamma_{\mathfrak{m}_r}(M)$. Therefore all results proved in Section 3 of [14] (in the case R is a local ring) can be extended for our notion of Σ-finite. For example, if M is a Σ-finite D-module, then $H^i_J(M)$ is also a Σ-finite D-module for every ideal $J \subset S$ and integer $i \geq 0$.

The following give us examples of Σ-finite D-modules.

Lemma 3.3. *Let A be an Artinian R-module. Then $M = A \otimes_R R[x_1, ..., x_n]$ (resp. $M = A \otimes_R R[[x_1, ..., x_n]]$) is a Σ-finite D-module.*

Proof. It is easy to see that $\mathrm{Supp}_R(M) = \mathrm{Supp}_R(A)$ is a finite subset of $\mathrm{Max}(R)$. Since A is Artinian, it is union of all submodules of finite length. Moreover if L is an R-module of finite length, then $L \otimes_R S$ is a D-module of finite length. The assertion now follows. □

Remark 3.4. Suppose that $S = R[[x_1, ..., x_n]]$. In general $A \otimes_R R[[x_1, ..., x_n]] \ncong A[[x_1, ..., x_n]]$ and $A[[x_1, ..., x_n]]$ may not be Σ-finite. For example, let $R = k[t]$, where k is a field and t an indeterminate. Let $A = E_R(k)$ be the injective hull of k. Then $A \cong k[t^{-1}]$. Choose the element $a = \sum_{i=0}^{\infty} t^{-i}x_1^i \in S$ we have $\mathrm{Ann}_R(a) = 0 \notin \mathrm{Max}(R)$.

Lemma 3.5. *Let I be an ideal of R such that $\dim R/I = 0$. Then $H^i_{IS}(S)$ is a Σ-finite D-module.*

Proof. We have $\dim R/I = 0$ so $\sqrt{I} = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_r$ with $\mathfrak{m}_i \in \mathrm{Max}(R)$ for all $i = 0, ..., r$. By the Mayer-Vietoris sequence we have $H^i_I(R) \cong H^i_{\mathfrak{m}_1}(R) \oplus \cdots \oplus H^i_{\mathfrak{m}_r}(R)$. So $H^i_I(R)$ is Artinian for all $i \geq 0$ by [3, Theorem 7.1.3]. By Lemma 3.3 we have $H^i_{IS}(S) \cong H^i_I(R) \otimes_R S$ is Σ-finite. □

The following is very useful in the sequel.

Lemma 3.6. *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of D-modules. Then*

(i) *If M is Σ-finite then M' and M'' are Σ-finite.*
(ii) *Conversely, if M' and M'' are Σ-finite and M' has finite length as a D-module, then M is Σ-finite.*

Proof. (i) This part is [14, Proposition 3.6].
(ii) Since M'' is Σ-finite we have $M'' = \cup_{N'' \in \mathrm{Fin}(M'')} N''$. For each $N'' \in \mathrm{Fin}(M'')$, let N be the preimage of N''. One can check that N admits a D-module structure. We have the following short exact sequence of D-modules.

$$0 \to M' \to N \to N'' \to 0.$$

Since M' has finite length as a D-module we have N has finite length as a D-module. Hence $M = \cup_{N \in \mathrm{Fin}(M)} N$. The two last conditions of Definition 3.1 are not difficult to prove. □

Recalling that a Serre's category is a category that closes under taking submodules, quotients and extensions. If R contains the rational numbers, then the category of Σ-finite D-modules is a Serre's subcategory of the category of D-module (cf. [14, Proposition 3.7]). At the time of writing, we do not know whether the condition $\mathbb{Q} \subseteq R$ can be removed. Fortunately, the statement of Lemma 3.6 (ii) is enough for our purpose. In the following we prove the global case of [14, Proposition 4.3]. While the proof of [14] is based on spectral sequences, our proof is elementary.

Proposition 3.7. *Let R be a (not necessary local) Noetherian ring and $S = R[x_1, ..., x_n]$ or $S = R[[x_1, ..., x_n]]$. Let J be an ideal of S such that $\dim R/J \cap R = 0$. Then $H^i_J(S)$ is Σ-finite for every $i \in \mathbb{N}$.*

Proof. We can assume that J is a radical ideal, so $J \cap R = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_r$ where $\mathfrak{m}_k \in \mathrm{Max}(R)$ for all $k = 1, ..., r$. Set $J_k = \mathfrak{m}_k S + J$, $k = 1, ..., r$, we have $J = J_1 \cap \cdots \cap J_r$. Since $\mathfrak{m}_k + \mathfrak{m}_h = R$ for all $k \neq h$, we have $J_k + J_h = S$ for all $k \neq h$. By using Mayer-Vietoris's sequence one can prove that

$$H^i_J(S) \cong H^i_{J_1}(S) \oplus \cdots \oplus H^i_{J_r}(S)$$

for all $i \geq 0$. Therefore, it is enough to prove the assertion in the case $J \cap R = \mathfrak{m} \in \mathrm{Max}(R)$ (cf. [14, Lemma 3.9]). We proceed by induction of $t = \mathrm{ht}(\mathfrak{m})$.

The case $t = 0$, we have that $\mathfrak{m}$ is a minimal prime of R. Let $U = H^0_{\mathfrak{m}}(R)$ and $\overline{R} = R/U$. We have U is an R-module of finite length so $U \otimes_R S$ is a Σ-finite D-module by Lemma 3.3.

By [14, Corollary 3.10], $H^i_J(U \otimes_R S)$ is Σ-finite for all $i \geq 0$. Applying local cohomology functor for the short exact sequence

$$0 \to U \otimes_R S \to S \to \overline{S} \to 0,$$

where $\overline{S} = \overline{R} \otimes_R S$, we get the following exact sequence

$$\cdots \to H^{i-1}_J(\overline{S}) \to H^i_J(U \otimes_R S) \to H^i_J(S) \to H^i_J(\overline{S}) \to \cdots.$$

On the other hand, we have $\mathrm{Ass}_R\overline{R} = \mathrm{Ass}_R R \setminus V(\mathfrak{m})$. Notice that $\mathrm{ht}(\mathfrak{m}) = 0$ so $\mathfrak{p} \nsubseteq \mathfrak{m}$ for all $\mathfrak{p} \in \mathrm{Ass}_R\overline{R}$, and hence $\mathrm{Ann}_R(\overline{R}) \nsubseteq \mathfrak{m}$. Moreover $\mathfrak{m} \in \mathrm{Max}(R)$ we have $\mathrm{Ann}_R(\overline{R}) + \mathfrak{m} = R$. Therefore $1 \in \mathrm{Ann}_R(\overline{R})S + J$ because $J \cap R = \mathfrak{m}$. Thus $\mathrm{Ann}_S(\overline{S}) + J = S$ since $\mathrm{Ann}_S(\overline{S}) = \mathrm{Ann}_R(\overline{R})S$. So $H^i_J(\overline{S}) = 0$ for all $i \geq 0$ and hence $H^i_J(S) \cong H^i_J(U \otimes_R S)$ is Σ-finite for all $i \geq 0$.

For $t > 0$, set $U = H^0_{\mathfrak{m}}(R)$ and $\overline{R} = R/H^0_{\mathfrak{m}}(R)$. Let $\overline{S} = \overline{R} \otimes_R S$. The short exact sequence

$$0 \to U \otimes_R S \to S \to \overline{S} \to 0$$

induces the exact sequence of local cohomology modules

$$\cdots \to H^i_J(U \otimes_R S) \xrightarrow{\alpha} H^i_J(S) \xrightarrow{\beta} H^i_J(\overline{S}) \to \cdots.$$

We have the short exact sequence

$$0 \to \mathrm{im}(\alpha) \to H^i_J(S) \to \mathrm{im}(\beta) \to 0.$$

Since U has finite length as an R-module, $U \otimes_R S$ and hence $H^i_J(U \otimes_R S)$ have finite length as a D-module by Example 2.1 (iv) (see also [12, Proposition 3.3]). Thus $\mathrm{im}(\alpha)$ is a D-module of finite length. Suppose $H^i_J(\overline{S})$ is Σ-finite we have $\mathrm{im}(\beta)$ is also a Σ-finite D-module by Lemma 3.6 (i). Lemma 3.6 (ii) implies that $H^i_J(S)$ is Σ-finite for all $i \geq 0$. Therefore we can assume henceforth that $H^0_{\mathfrak{m}}(R) = 0$. Choose an R-regular element $a \in \mathfrak{m} = J \cap R$, we have a is also S-regular and $a \in J$. So $H^0_J(S) = 0$. For $i \geq 1$ we consider the following short exact sequence

$$0 \to S \to S_a \to S_a/S \to 0.$$

This sequence induces the exact sequence of local cohomology

$$\cdots \to H^{i-1}_J(S_a) \to H^{i-1}_J(S_a/S) \to H^i_J(S) \to H^i_J(S_a) \to \cdots.$$

Notice that $a \in J$, so $H^i_J(S_a) = 0$ for all $i \geq 0$. Thus

$$H^i_J(S) \cong H^{i-1}_J(S_a/S) \cong H^{i-1}_J(\varinjlim_n (S/a^nS)) \cong \varinjlim_n H^{i-1}_J(S/a^nS).$$

By inductive hypothesis we have $H^{i-1}_J(S/a^nS)$ is a Σ-finite $D(S/a^nS, R/a^nR)$-module for all n and $i \geq 1$. So $H^{i-1}_J(S/a^nS)$ is a Σ-finite $D(S,R)$-module for all n and $i \geq 1$. By [14, Proposition 3.11] we need only to prove that $\cup_n \mathcal{C}(H^i_J(S/a^nS)$ is finite for all $i \geq 0$. We shall prove that $\mathcal{C}(H^i_J(S/a^nS) \subseteq \mathcal{C}(H^i_J(S/aS))$ for all $n \geq 1$. The case $n = 1$ is trivial. For $n > 1$, the short exact sequence

$$0 \to S/aS \xrightarrow{a^{n-1}\cdot} S/a^nS \to S/a^{n-1}S \to 0$$

induces the exact sequence

$$\cdots \to H^i_J(S/aS) \to H^i_J(S/a^nS) \to H^i_J(S/a^{n-1}S) \to \cdots.$$

Hence $\mathcal{C}(H^i_J(S/a^nS)) \subseteq \mathcal{C}(H^i_J(S/aS)) \cup \mathcal{C}(H^i_J(S/a^{n-1}S)) \subseteq \mathcal{C}(H^i_J(S/aS))$ by inductive hypothesis. The proof is complete. □

We are ready to prove the main result of this section, it gives a partial positive answer for [14, Question 5.1].

Theorem 3.8. *Let $(R, \mathfrak{m})$ be a local ring and $S = R[x_1, ..., x_n]$ or $S = R[[x_1, ..., x_n]]$. Let J be an ideal of S such that $\dim R/(J \cap R) \leq 1$. Then $H^j_{\mathfrak{m}S} H^i_J(S)$ is Σ-finite for every $i, j \in \mathbb{N}$. In particular $\mathrm{Ass}_S H^j_{\mathfrak{m}S} H^i_J(S)$ is finite for all $i, j \in \mathbb{N}$.*

Proof. Since $\dim R/(J \cap R) \leq 1$, there exists $f \in \mathfrak{m}$ such that $\mathfrak{m}S \subset \sqrt{(J + fS)}$. Thus $\sqrt{J + \mathfrak{m}S} = \sqrt{J + fS}$. Notice that $H^i_J(S)$ is J-torsion. So

$$H^j_{\mathfrak{m}S} H^i_J(S) \cong H^j_{(J+\mathfrak{m}S)} H^i_J(S) \cong H^j_{(J+fS)} H^i_J(S) \cong H^j_{fS} H^i_J(S)$$

for all $i, j \geq 0$. Therefore $H^j_{\mathfrak{m}S} H^i_J(S) = 0$ for all $j > 1$. Hence we need only to prove that $H^0_{fS} H^i_J(S)$ and $H^1_{fS} H^i_J(S)$ are Σ-finite for all $i \geq 0$. By [3, Proposition 8.1.2] we have the following exact sequence

$$\cdots \to H^{i-1}_J(S) \to H^{i-1}_J(S_f) \to H^i_{(J+fS)}(S) \to H^i_J(S) \to H^i_J(S_f) \to \cdots .$$

On the other hand we have the following exact sequence (cf. [3, Remark 2.2.17])

$$0 \to H^0_{fS} H^i_J(S) \to H^i_J(S) \to H^i_J(S_f) \to H^1_{fS} H^i_J(S) \to 0$$

for all $i \geq 0$. Therefore for each $i \geq 0$ we have the following short exact sequence

$$0 \to H^1_{fS} H^{i-1}_J(S) \to H^i_{(J+fS)}(S) \to H^0_{fS} H^i_J(S) \to 0.$$

Since $\dim R/((J + fS) \cap R) = 0$, we have $H^i_{(J+fS)}(S)$ is Σ-finite for all $i \geq 0$ by Proposition 3.7. Hence $H^0_{fS} H^i_J(S)$ and $H^1_{fS} H^i_J(S)$ are Σ-finite for all $i \geq 0$ by Lemma 3.6. The last assertion follows from the property of Σ-finite D-modules. The proof is complete. □

We get a result of on the finiteness of associated primes of local cohomology of polynomial rings.

Corollary 3.9. *Let $(R, \mathfrak{m})$ be a local ring and $S = R[x_1, ..., x_n]$. Let J be an ideal of S such that $\dim R/(J \cap R) \leq 1$. Then $\mathrm{Ass}_S H^i_J(S)$ is finite for all $i \geq 0$.*

Proof. Similarly the proof of Theorem 3.8 we have an element $f \in \mathfrak{m}$ such that $\mathfrak{m}S \subseteq \sqrt{(J + fS)}$. Consider the exact sequence

$$\cdots \to H^i_{(J+fS)}(S) \stackrel{\alpha}{\to} H^i_J(S) \to H^i_J(S_f) \to \cdots .$$

We have $\mathrm{Ass}_S H^i_J(S) \subseteq \mathrm{Ass}_S(\mathrm{im}(\alpha)) \cup \mathrm{Ass}_S H^i_J(S_f)$. Since $H^i_{(J+fS)}(S)$ is Σ-finite, so is $\mathrm{im}(\alpha)$. Hence $\mathrm{Ass}_S(\mathrm{im}(\alpha))$ is a finite set. On the other hand we have $H^i_J(S_f) \cong H^i_{(JS_f)}(S_f)$. Notice that $S_f \cong R_f[x_1, ..., x_n]$ and $\dim R_f/(JS_f \cap R_f) = 0$, we have $H^i_J(S_f)$ is a Σ-finite $D(S_f, R_f)$-module by Proposition 3.7. So $\mathrm{Ass}_S H^i_J(S_f)$ is finite. The proof is complete. □

4. Rings of dimension one

In this section R is a Noetherian ring of dimension one and $S = R[x_1, ..., x_n]$ or $S = R[[x_1, ..., x_n]]$. We recall our question.

Question 3. Is it true that $H^i_J(S)$ has only finitely many associated primes for all ideals J of S and all $i \geq 0$?

The following is an immediate consequence of Corollary 3.9 which was shown before by Núñez-Betancourt in [12, Corollary 3.7].

Corollary 4.1. *Suppose that R is local and $S = R[x_1, ..., x_n]$. Then $\mathrm{Ass}_S H^i_J(S)$ is finite for all ideal J and all $i \geq 0$.*

We shall consider the question when R contains a field of characteristic $p > 0$. We start with the following.

Lemma 4.2. *Let W is the largest ideal of finite length of R and $\overline{R} = R/W$. Let $\overline{S} = \overline{R} \otimes_R S$. Suppose $\mathrm{Ass}_S H^i_J(\overline{S})$ is finite for all $i \geq 0$. Then $\mathrm{Ass}_S H^i_J(S)$ is finite for all $i \geq 0$.*

Proof. The short exact sequence

$$0 \to W \otimes_R S \to S \to \overline{S} \to 0$$

induces the exact sequence of local cohomology

$$\cdots \to H^i_J(W \otimes_R S) \xrightarrow{\alpha} H^i_J(S) \to H^i_J(\overline{S}) \to \cdots .$$

Since W has finite length we have $H^i_J(W \otimes_R S)$ is a Σ-finite D-module by Lemma 3.3 and Remark 3.2. Hence so is $\mathrm{im}(\alpha)$. Moreover $\mathrm{Ass}_S H^i_J(S) \subseteq \mathrm{Ass}_S(\mathrm{im}(\alpha)) \cup \mathrm{Ass}_S H^i_J(\overline{S})$. Therefore if $\mathrm{Ass}_S H^i_J(\overline{S})$ is finite, then so is $\mathrm{Ass}_S H^i_J(S)$. □

Proposition 4.3. *Let R be an excellent domain of dimension one and of characteristic $p > 0$. Then $\mathrm{Ass}_S H^i_J(S)$ is finite for all ideal J and all $i \geq 0$.*

Proof. Let T be the integral closure of R. We have T is a finitely generated R-module. Since $\dim R = 1$ we have T/R is an R-module of finite length. Set $V = T \otimes_R S$. Then V is either $T[x_1, ..., x_n]$ or $T[[x_1, ..., x_n]]$. The short exact sequence

$$0 \to S \to V \to V/S \to 0$$

induces the exact sequence

$$\cdots \to H^{i-1}_J(V/S) \xrightarrow{\alpha} H^i_J(S) \to H^i_J(V) \to \cdots .$$

Notice that V/S is a Σ-finite D-module of finite length and so is $H^{i-1}_J(V/S)$. Therefore $\mathrm{Ass}_S(\mathrm{im}(\alpha))$ is finite. Since T is Dedekind we have V is a regular ring of characteristic $p > 0$. So $\mathrm{Ass}_V H^i_{JV}(V)$ is finite by [7] or [9]. By the independent theorem we have $H^i_J(V) \cong H^i_{JV}(V)$. Thus $\mathrm{Ass}_S H^i_J(V)$ is finite. The proof is complete. □

The following is the main result of this section.

Proposition 4.4. *Let R be an excellent reduced ring of dimension one and of characteristic $p > 0$. Let S is either $R[x_1, ..., x_n]$ or $R[[x_1, ..., x_n]]$. Then $\mathrm{Ass}_S H^i_J(S)$ is finite for all ideal J and all $i \geq 0$.*

Proof. By Lemma 4.2 we can assume that $\dim R/\mathfrak{p} = 1$ for all $\mathfrak{p} \in \mathrm{Ass}_R R$. Since R is reduced, $0 = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$. We proceed by induction on r. The case $r = 1$ follows from Proposition 4.3. For $r > 1$, the following exact sequence

$$0 \to S \to (S/(\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_{r-1})S) \oplus S/\mathfrak{p}_r S \to S/(\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_{r-1} + \mathfrak{p}_r)S \to 0$$

induces the exact sequence

$$\cdots \to H^{i-1}_J(S/(\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_{r-1} + \mathfrak{p}_r)S) \xrightarrow{\alpha} H^i_J(S) \to H^i_J(S/(\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_{r-1})S) \oplus H^i_J(S/\mathfrak{p}_r S) \to \cdots.$$

Since $\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_{r-1} + \mathfrak{p}_r$ is not contained in any minimal prime and $\dim R = 1$, we have $R/(\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_{r-1} + \mathfrak{p}_r)$ has finite length. Thus

$$H^{i-1}_J(S/(\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_{r-1} + \mathfrak{p}_r)S)$$

is Σ-finite for all $i \geq 1$. Thus $\mathrm{Ass}_S(\mathrm{im}(\alpha))$ is finite. Combining with the inductive hypothesis we obtain the assertion. □

Inspired by [1] and [2] we raise the following question.

Question 4. *Let R be a Noetherian ring of dimension zero and of characteristic $p > 0$. Let $S = R[x_1, ..., x_n]$ or $S = R[[x_1, ..., x_n]]$. For each ideal $J = (a_1, ..., a_t)$ of S, is it true that the image of the canonical map*

$$\varphi : H^i(a_1, ..., a_t; S) \to H^i_J(S)$$

generates $H^i_J(S)$ as a D-module.

If the above question has a positive answer, then by the same method used in [2] we can extend the result of Proposition 4.4 in the case $S = R[x_1, ..., x_n]$ for any ring of dimension one and of characteristic $p > 0$.

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