ON THE VANISHING OF LOCAL COHOMOLOGY OF THE ABSOLUTE INTEGRAL CLOSURE IN POSITIVE CHARACTERISTIC

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ABSTRACT. The aim of this paper is to extend the main result of C. Huneke and G. Lyubeznik in [Adv. Math. 210 (2007), 498–504] to the class of rings that are images of Cohen-Macaulay local rings. Namely, let R be a local Noetherian domain of positive characteristic that is an image of a Cohen-Macaulay local ring. We prove that all local cohomology of R (below the dimension) maps to zero in a finite extension of the ring. As a direct consequence we obtain that the absolute integral closure of R is a big Cohen-Macaulay algebra. Since every excellent local ring is an image of a Cohen-Macaulay local ring, this result is a generalization of the main result of M. Hochster and Huneke in [Ann. of Math. 135 (1992), 45–79] with a simpler proof.

1. INTRODUCTION

Let $(R, \mathfrak{m})$ be a commutative Noetherian local domain with fraction field K. The *absolute integral closure* of R, denoted R^+, is the integral closure of R in a fixed algebraic closure $\overline{K}$ of K.

A famous result of M. Hochster and C. Huneke says that if $(R, \mathfrak{m})$ is an excellent local Noetherian domain of positive characteristic $p > 0$, then R^+ is a (balanced) big Cohen-Macaulay algebra, i.e. every system of parameters in R becomes a regular sequence in R^+ (cf. [7]). Furthermore, K.E. Smith in [15] proved that the tight closure of an ideal generated by parameters is the contraction of its extension in R^+: $I^* = IR^+ \cap R$. This property is not true for every ideal I in an excellent Noetherian domain since tight closure does not commute with localization (cf. [1]).

As mentioned above, $H^i_{\mathfrak{m}}(R^+) = 0$ for all $i < \dim R$ provided R is an excellent local Noetherian domain of positive characteristic. Hence, the natural homomorphism $H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(R^+)$, induced from the inclusion $R \to R^+$, is the zero map for all $i < \dim R$. In the case R is an image of a Gorenstein (not necessarily excellent) local ring, as the main result of [8], Huneke and G. Lyubeznik proved a stronger conclusion that one can find a finite extension ring S, $R \subseteq S \subseteq R^+$, such that the natural map $H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(S)$ is zero for all $i < \dim R$. Therefore, they obtained a simpler proof for the result of Hochster and Huneke in the cases where the assumptions overlap, e.g., for complete Noetherian local domain. The techniques used in [8] are the Frobenius action on the local cohomology, (modified) equation lemma (cf. [7], [15], [8]) and the local duality theorem (This is the reason of the assumption that R is an image of a Gorenstein local ring). The motivation of the present paper is our belief: *If a result was shown by the local duality theorem, then it can be proven under the assumption that the ring is an image of a Cohen-Macaulay local ring* (for example, see [12]). The main result of this paper extends Huneke-Lyubeznik's result to the class of rings that are images of Cohen-Macaulay local rings. Namely, we prove the following.

Key words and phrases. Absolute integral closure; Local cohomology; Big Cohen-Macaulay; Characteristics p.

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Theorem 1.1. *Let $(R, \mathfrak{m})$ be a commutative Noetherian local domain containing a filed of positive characteristic p. Let K be the fraction field of R and $\overline{K}$ an algebraic closure of K. Assume that R is an image of a Cohen-Macaulay local ring. Let R' be an R-subalgebra of $\overline{K}$ (i.e. $R \subseteq R' \subseteq \overline{K}$) that is a finite R-module. Then there is an R'-subalgebra R'' of $\overline{K}$ (i.e. $R' \subseteq R'' \subseteq \overline{K}$) that is finite as an R-module such that the natural map $H^i_{\mathfrak{m}}(R') \to H^i_{\mathfrak{m}}(R'')$ is the zero map for all $i < \dim R$.*

As a direct application of Theorem 1.1 we obtain that the absolute integral closure R^+ is a big Cohen-Macaulay algebra (cf. Corollary 3.2). It worth be noted that every excellent local ring is an image of a Cohen-Macaulay excellent local ring by T. Kawasaki (cf. [9, Corollary 1.2]). Therefore, our results also extend the original result of Hochster and Huneke with a simpler proof. The main results will be proven in the last section. In the next section we recall the theory of attached primes of Artinian (local cohomology) modules.

2. Preliminaries

Throughout this section $(R, \mathfrak{m})$ be a commutative Noetherian local ring. We recall the main result of [12] which is an illustration for our belief (mentioned in the introduction).

I.G. Macdonald, in [10], introduced the theory of secondary representation for Artinian modules, which is in some sense dual to the theory of primary decomposition for Noetherian modules. Let $A \neq 0$ be an Artinian R-module. We say that A is *secondary* if the multiplication by x on A is surjective or nilpotent for every $x \in R$. In this case, the set $\mathfrak{p} := \sqrt{(\mathrm{Ann}_R A)}$ is a prime ideal of R and we say that A is $\mathfrak{p}$-*secondary*. Note that every Artinian R-module A has a minimal secondary representation $A = A_1 + \ldots + A_n$, where A_i is $\mathfrak{p}_i$-secondary, each A_i is not redundant and $\mathfrak{p}_i \neq \mathfrak{p}_j$ for all $i \neq j$. The set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ is independent of the choice of the minimal secondary representation of A. This set is called the set of *attached primes* of A and denoted by $\mathrm{Att}_R A$. Notice that if R is complete we have the Matlis dual $D(A)$ of A is Noetherian and $\mathrm{Att}_R A = \mathrm{Ass}_R D(A)$.

For each ideal I of R, we denote by $Var(I)$ the set of all prime ideals of R containing I. The following is easy to understand from the theory of associated primes.

Lemma 2.1 ([10])**.** *Let A be an Artinian R-module. The following statements are true.*

(i) $A \neq 0$ *if and only if* $\mathrm{Att}_R A \neq \emptyset$.
(ii) $A \neq 0$ *has finite length if and only if* $\mathrm{Att}_R A \neq \{\mathfrak{m}\}$.
(iii) $\min \mathrm{Att}_R A = \min Var(\mathrm{Ann}_R A)$. *In particular,*

$$\dim(R/\mathrm{Ann}_R A) = \max\{\dim(R/\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Att}_R A\}.$$

(iv) *If* $0 \to A' \to A \to A'' \to 0$ *is an exact sequence of Artinian R-modules then*

$$\mathrm{Att}_R A'' \subseteq \mathrm{Att}_R A \subseteq \mathrm{Att}_R A' \cup \mathrm{Att}_R A''.$$

Let $\widehat{R}$ be the $\mathfrak{m}$-adic complete of R. Note that every Artinan R-module A has a natural structure as an $\widehat{R}$-module and with this structure, each subset of A is an R-submodule if and only if it is an $\widehat{R}$-submodule. Therefore A is an Artinian $\widehat{R}$-module. So, the set of attached primes $\mathrm{Att}_{\widehat{R}} A$ of A over $\widehat{R}$ is well defined.

Lemma 2.2. *(*[2, 8.2.4, 8.2.5]*).* $\mathrm{Att}_R A = \big\{P \cap R \mid P \in \mathrm{Att}_{\widehat{R}} A\big\}$.

Let M be a finitely generated R-module. It is well known that the local cohomology module $H^i_{\mathfrak{m}}(M)$ is Artinian for all $i \geq 0$ (cf. [2, Theorem 7.1.3]). Suppose that R is an image of a Gorenstein local ring. R.Y. Sharp, in [14], used the local duality theorem to prove the following relation

$$\mathrm{Att}_{R_{\mathfrak{p}}}\big(H^{i-\dim(R/\mathfrak{p})}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}})\big) = \big\{\mathfrak{q}R_{\mathfrak{p}} \mid \mathfrak{q} \in \mathrm{Att}_R(H^i_{\mathfrak{m}}(M)), \mathfrak{q} \subseteq \mathfrak{p}\big\}$$

for all $\mathfrak{p} \in \operatorname{Supp}(M)$ and all $i \geq 0$. Based on the study of splitting of local cohomology (cf. [4], [5]), L.T. Nhan and the author showed that the above relation holds true on the category of finitely generated R-modules if and only if R is an image of a Cohen-Macaulay local ring (cf. [12]). It worth be noted that R is an image of a Cohen-Macaulay local ring if and only if R is universally catenary and all its formal fibers are Cohen-Macaulay by Kawasaki (cf. [9, Corollary 1.2]). More precisely, we proved the following.

Theorem 2.3. *The following statements are equivalent:*

(i) *R is an image of a Cohen-Macaulay local ring;*

(ii) $\operatorname{Att}_{R_\mathfrak{p}}\big(H^{i-\dim(R/\mathfrak{p})}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})\big) = \big\{\mathfrak{q}R_\mathfrak{p} \mid \mathfrak{q} \in \operatorname{Att}_R(H^i_\mathfrak{m}(M)), \mathfrak{q} \subseteq \mathfrak{p}\big\}$ *for every finitely generated R-module M, integer $i \geq 0$ and prime ideal $\mathfrak{p}$ of R;*

(iii) $\operatorname{Att}_{\widehat{R}}(H^i_\mathfrak{m}(M)) = \displaystyle\bigcup_{\mathfrak{p} \in \operatorname{Att}_R(H^i_\mathfrak{m}(M))} \operatorname{Ass}_{\widehat{R}}(\widehat{R}/\mathfrak{p}\widehat{R})$ *for every finitely generated R-module M and integer $i \geq 0$.*

The above Theorem says that the attached primes of local cohomology modules have good behaviors with completion and localization when R is an image of a Cohen-Macaulay local ring. This will be very useful in the next section.

3. Proof the main result

Throughout this section, let $(R, \mathfrak{m}, k)$ be a commutative Noetherian local ring that is an image of a Cohen-Macaulay local ring. The following plays the key role in our proof of the main result.

Proposition 3.1. *Let M and N be finitely generated R-modules and $\varphi : M \to N$ a homomorphism. For each $i \geq 0$, φ induces the homomorphism $\varphi^i : H^i_\mathfrak{m}(M) \to H^i_\mathfrak{m}(N)$. Suppose for all $\mathfrak{p} \in \operatorname{Att}_R(H^i_\mathfrak{m}(M))$ and $\mathfrak{p} \neq \mathfrak{m}$, the map $\varphi_\mathfrak{p} : M_\mathfrak{p} \to N_\mathfrak{p}$ induces the zero map*

$$\varphi_\mathfrak{p}^{i-t_\mathfrak{p}} : H^{i-t_\mathfrak{p}}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p}) \to H^{i-t_\mathfrak{p}}_{\mathfrak{p}R_\mathfrak{p}}(N_\mathfrak{p}),$$

where $t_\mathfrak{p} = \dim R/\mathfrak{p}$. Then $\operatorname{Im}(\varphi^i)$ has finite length.

Proof. Suppose $\operatorname{Im}(\varphi^i)$ has not finite length. By Lemma 2.1 there exists $\mathfrak{m} \neq \mathfrak{p} \in \operatorname{Att}_R(\operatorname{Im}(\varphi^i))$. So $\mathfrak{p} \in \operatorname{Att}_R(H^i_\mathfrak{m}(M))$ by Lemma 2.1 (iv). Consider $\operatorname{Im}(\varphi^i)$ as an Artinian $\widehat{R}$-module. By Lemma 2.2, there exists $P \in \operatorname{Att}_{\widehat{R}}(\operatorname{Im}(\varphi^i))$ such that $P \cap R = \mathfrak{p}$. Hence we have $P \in \operatorname{Att}_{\widehat{R}}(H^i_\mathfrak{m}(M))$ by Lemma 2.1 (iv) again. Since R is an image of a Cohen-Macaulay local ring, Theorem 2.3 (iii) implies that $P \in \operatorname{Ass}_{\widehat{R}}(\widehat{R}/\mathfrak{p}\widehat{R})$. Therefore $\dim \widehat{R}/P = \dim R/\mathfrak{p}$ by [3, Theorem 2.1.15]. We have $\widehat{R}$ is complete, so it is an image of a Gorenstein local ring S (of dimension n). By local duality we have

$$D(\operatorname{Ext}^{n-i}_S(\widehat{M}, S)) \cong H^i_{\widehat{\mathfrak{m}}}(\widehat{M}) \quad (\cong H^i_\mathfrak{m}(M) \otimes_R \widehat{R} \cong H^i_\mathfrak{m}(M)),$$

where $D = \operatorname{Hom}_{\widehat{R}}(-, E_{\widehat{R}}(k))$ is the Matlis duality functor (cf. [2, Theorem 11.2.6]). Since $\widehat{R}$ is complete we have $\operatorname{Ext}^{n-i}_S(\widehat{M}, S) \cong D(H^i_\mathfrak{m}(M))$.
We write the map $\varphi^i : H^i_\mathfrak{m}(M) \to H^i_\mathfrak{m}(N)$ as the composition of two maps

$$H^i_\mathfrak{m}(M) \to \mathcal{I} = \operatorname{Im}(\varphi^i) \to H^i_\mathfrak{m}(N),$$

where the first of which is surjective and the second injective. Applying the Matlis duality functor D we get the map $D(\varphi^i) : \operatorname{Ext}^{n-i}_S(\widehat{N}, S) \to \operatorname{Ext}^{n-i}_S(\widehat{M}, S)$ is the composition of two maps

$$\operatorname{Ext}^{n-i}_S(\widehat{N}, S) \to D(\mathcal{I}) \to \operatorname{Ext}^{n-i}_S(\widehat{M}, S)$$

with the first of which is surjective and the second injective. We have $D(\mathcal{I})$ is a finitely generated $\widehat{R}$-module and $P \in \operatorname{Ass}_{\widehat{R}} D(\mathcal{I})$ $(= \operatorname{Att}_{\widehat{R}}\mathcal{I})$. Let P' be the pre-image of P in S. Localization at P' the above composition we get the composition

$$\operatorname{Ext}^{n-i}_{S_{P'}}(\widehat{N}_P, S_{P'}) \to (D(\mathcal{I}))_P \to \operatorname{Ext}^{n-i}_{S_{P'}}(\widehat{M}_P, S_{P'})$$

with the first of which is surjective and the second injective. Since $(D(\mathcal{I}))_P \neq 0$, we have the map

$$D(\varphi^i)_P : \operatorname{Ext}^{n-i}_{S_{P'}}(\widehat{N}_P, S_{P'}) \to \operatorname{Ext}^{n-i}_{S_{P'}}(\widehat{M}_P, S_{P'})$$

is a non-zero map. Notice that $\dim S_{P'} = n - t_{\mathfrak{p}}$. Applying local duality (for $S_{P'}$) we have the map

$$\widehat{\varphi}_P^{i-t_{\mathfrak{p}}} : H^{i-t_{\mathfrak{p}}}_{P\widehat{R}_P}(\widehat{M}_P) \to H^{i-t_{\mathfrak{p}}}_{P\widehat{R}_P}(\widehat{N}_P),$$

induced from the map $\widehat{\varphi} : \widehat{M} \to \widehat{N}$, is a non-zero map. Recalling our assumption that the map

$$\varphi_{\mathfrak{p}}^{i-t_{\mathfrak{p}}} : H^{i-t_{\mathfrak{p}}}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \to H^{i-t_{\mathfrak{p}}}_{\mathfrak{p}R_{\mathfrak{p}}}(N_{\mathfrak{p}}),$$

induced from $\varphi : M \to N$, is zero.

On the other hand, the faithfully flat homomorphism of local rings $(R, \mathfrak{m}) \to (\widehat{R}, \widehat{\mathfrak{m}})$ induces the flat homomorphism $(R_{\mathfrak{p}}, \mathfrak{p}R_{\mathfrak{p}}) \to (\widehat{R}_P, P\widehat{R}_P)$ by [11, Theorem 7.1]. It is a local homomorphism so we have a faithfully flat homomorphism. It should be noted that $\sqrt{\mathfrak{p}\widehat{R}_P} = P\widehat{R}_P$. Using the following commutative diagram of flat homomorphisms

$$\begin{array}{ccc} R & \longrightarrow & R_{\mathfrak{p}} \\ \downarrow & & \downarrow \\ \widehat{R} & \longrightarrow & \widehat{R}_P \end{array}$$

one can check that

$$H^{i-t_{\mathfrak{p}}}_{P\widehat{R}_P}(\widehat{M}_P) \cong H^{i-t_{\mathfrak{p}}}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \otimes_{R_{\mathfrak{p}}} \widehat{R}_P$$

and $\widehat{\varphi}_P^{i-t_{\mathfrak{p}}}$ is just $\varphi_{\mathfrak{p}}^{i-t_{\mathfrak{p}}} \otimes_{R_{\mathfrak{p}}} \widehat{R}_P$. Therefore the maps $\varphi_{\mathfrak{p}}^{i-t_{\mathfrak{p}}}$ and $\widehat{\varphi}_P^{i-t_{\mathfrak{p}}}$ are either zero or non-zero, simultaneously. This is a contradiction. The proof is complete. □

We are ready to prove the main result of this paper. In the rest of this section, $(R, \mathfrak{m})$ is a local domain of positive characteristic p that is an image of a Cohen-Macaulay local ring. Let I be an ideal of R and R' an R-algebra. Notice that the local cohomology, $H^i_I(-)$, can be computed via the Čech co-complex of the generators of I. The Frobenius ring homomorphism

$$f : R' \longrightarrow R'; r \mapsto r^p$$

induces a natural map $f_* : H^i_I(R') \to H^i_I(R')$ on all $i \geq 0$. It is called the (natural) action of Frobenius on $H^i_I(R')$.

Proof of Theorem 1.1. We proceed by induction on $d = \dim R$. There is nothing to prove when $d = 0$. Assume that $d > 0$ and the theorem is proven for all smaller dimension. For each $i < d$ and $\mathfrak{p} \in \operatorname{Att}_R H^i_{\mathfrak{m}}(R)$, $\mathfrak{p} \neq \mathfrak{m}$, by the inductive hypothesis there is an $R'_{\mathfrak{p}}$-subalgebra $\widetilde{R}^{i,\mathfrak{p}}$ that is finite as $R_{\mathfrak{p}}$-module such that the natural map

$$H^{i-t_{\mathfrak{p}}}_{\mathfrak{p}R_{\mathfrak{p}}}(R'_{\mathfrak{p}}) \to H^{i-t_{\mathfrak{p}}}_{\mathfrak{p}R_{\mathfrak{p}}}(\widetilde{R}^{i,\mathfrak{p}})$$

is the zero map, where $t_{\mathfrak{p}} = \dim R/\mathfrak{p}$. Let $\widetilde{R}^{i,\mathfrak{p}} = R'_{\mathfrak{p}}[z_1,...,z_k]$, where $z_1,\ldots,z_k \in \overline{K}$ are integral over $R_{\mathfrak{p}}$. Multiplying, if necessary, some suitable element of $R \setminus \mathfrak{p}$, we can assume that each z_j is integral over R. Set $R^{i,\mathfrak{p}} = R'[z_1,\ldots,z_k]$. Clearly, $R^{i,\mathfrak{p}}$ is an R'-subalgebra of $\overline{K}$ that is finite as R-module.
Since the sets $\{i \mid 0 \le i < d\}$ and $\mathrm{Att}_R(H^i_{\mathfrak{m}}(R))$ are finite, the following is a finite extension of R

$$R^* = R[R^{i,\mathfrak{p}} \mid i < d, \mathfrak{p} \in \mathrm{Att}_R(H^i_{\mathfrak{m}}(R)) \setminus \{\mathfrak{m}\}].$$

We have R^* is an $R^{i,\mathfrak{p}}$-subalgebra of $\overline{K}$ for all $i < d$ and all $\mathfrak{p} \in \mathrm{Att}_R(H^i_{\mathfrak{m}}(R)) \setminus \{\mathfrak{m}\}$. The inclusions $R' \to R^{i,\mathfrak{p}} \to R^*$ induce the natural maps

$$H^{i-t_{\mathfrak{p}}}_{\mathfrak{p}R_{\mathfrak{p}}}(R'_{\mathfrak{p}}) \to H^{i-t_{\mathfrak{p}}}_{\mathfrak{p}R_{\mathfrak{p}}}(\widetilde{R}^{i,\mathfrak{p}}) \to H^{i-t_{\mathfrak{p}}}_{\mathfrak{p}R_{\mathfrak{p}}}(R^*_{\mathfrak{p}}).$$

By the construction of $\widetilde{R}^{i,\mathfrak{p}}$ we have the natural map

$$H^{i-t_{\mathfrak{p}}}_{\mathfrak{p}R_{\mathfrak{p}}}(R'_{\mathfrak{p}}) \to H^{i-t_{\mathfrak{p}}}_{\mathfrak{p}R_{\mathfrak{p}}}(R^*_{\mathfrak{p}})$$

is the zero map for all $i < d$ and all $\mathfrak{p} \in \mathrm{Att}_R H^i_{\mathfrak{m}}(R) \setminus \{\mathfrak{m}\}$. By Proposition 3.1 we have the natural map

$$\varphi^i : H^i_{\mathfrak{m}}(R') \to H^i_{\mathfrak{m}}(R^*),$$

induced from the inclusion $\varphi : R' \to R^*$, has $\ell(\mathrm{Im}(\varphi^i)) < \infty$ for all $i < d$.

Since the natural inclusion $\varphi : R' \to R^*$ is compatible with the Frobenius homomorphism on R' and R^*, we have φ^i is compatible with the Frobenius action f_* on $H^i_{\mathfrak{m}}(R')$ and $H^i_{\mathfrak{m}}(R^*)$. Therefore $\mathrm{Im}\varphi^i$ is an f_*-stable R-submodule of $H^i_{\mathfrak{m}}(R^*)$, i.e. $f_*(\alpha) \in \mathrm{Im}\varphi^i$ for every $\alpha \in \mathrm{Im}\varphi^i$. Since $\mathrm{Im}\varphi^i$ has finite length, every $\alpha \in \mathrm{Im}\varphi^i$ satisfies the "equation lemma" of Huneke-Lyubeznik (cf. [8, Lemma 2.2]). Hence each element of $\mathrm{Im}\varphi^i$ will map to the zero in local cohomology of a certainly finite extension of R^*. Since $\mathrm{Im}\varphi^i$ is a finitely generated R-module for all $i < d$, there is an R^*-subalgebra R'' of $\overline{K}$ that is finite as R-module such that the composition of the natural maps $H^i_{\mathfrak{m}}(R') \to H^i_{\mathfrak{m}}(R^*) \to H^i_{\mathfrak{m}}(R'')$ is zero for all $i < d$. The proof is complete. □

Similar to [8, Corollary 2.3] we have the following.

Corollary 3.2. *Let $(R, \mathfrak{m})$ be a commutative Noetherian local domain containing a field of positive characteristic p and R^+ the absolute integral closure of R in $\overline{K}$. Assume that R is an image of a Cohen-Macaulay local ring. Then the following hold:*

(i) $H^i_{\mathfrak{m}}(R^+) = 0$ *for all* $i < \dim R$.
(ii) *Every system of parameters of R is a regular sequence on R^+, i.e. R^+ is a big Cohen-Macaulay algebra.*

We close this paper with the following.

Remark 3.3. (i) In [13], A. Sannai and A.K. Singh showed that the finite extension in Huneke-Lyubeznik's result can be chosen as a generically Galois extension. It is not difficult to see that our method also works for Sannai-Singh's paper. Hence Theorem 1.3 (2) and Corollary 3.3 of [13] hold true when the ring is an image of a Cohen-Macaulay local ring.
(ii) Since an excellent local ring is an image of a Cohen-Macaulay excellent local ring ([9, Corollary 1.2]), Corollary 3.3 is a generalization of the original result of Hochster and Huneke in [7] with a simpler proof. Thus our results cover all previous results. On the other hand, R.C. Heitmann constructed examples of universally catenarian local domains with the absolute integral closures are not Cohen-Macaulay (cf. [6, Corollary 1.8]). Comparing with the result

of Kawasaki ([9, Corollary 1.2]), the condition that R is an image of a Cohen-Macaulay local ring is seem to be the most general case for R^+ is big Cohen-Macaulay.

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