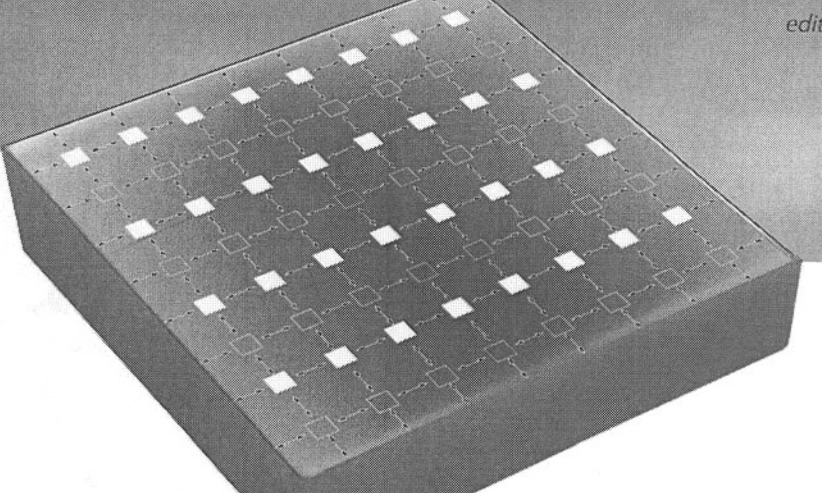
Series in Contemporary Applied Mathematics
CAM 8

Some Topics in Industrial and Applied Mathematics

Rolf Jeltsch
Ta-Tsien Li
Ian H Sloan
editors



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Preface

On the occasion that the Officers' Meeting and the Board Meeting of ICIAM (International Council for Industrial and Applied Mathematics) was held in Shanghai from May 26 to May 27, 2006, many famous industrial and applied mathematicians gathered in Shanghai from different countries. The Shanghai Forum on Industrial and Applied Mathematics was organized from May 25 to May 26, 2006 at Shanghai Science Hall for the purpose of inviting some of them to present their recent results and discuss recent trends in industrial and applied mathematics. Sixteen invited lectures have been given for this activity. This volume collects the material covered by most of these lectures. It will be very useful for graduate students and researchers in industrial and applied mathematics.

The editors would like take this opportunity to express their sincere thanks to all the authors in this volume for their kind contribution. We are very grateful to the Shanghai Association for Science and Technology (SAST), Fudan University, the National Natural Science Foundation of China (NSFC), The China Society for Industrial and Applied Mathematics (CSIAM), the Shanghai Society for Industrial and Applied Mathmatics (SSIAM), the Institut Sino-Français de Mathématiques Appliquées (ISFMA) and the International Council for Industrial and Applied Mathematics (ICIAM) for their help and support. Our special thanks are also due to Mrs. Zhou Chunlian for her efficient assistance in editing this book.

Rolf Jeltsch, Ta-Tsien Li, Ian H. Sloan
April 2007

Contents

Complementarity Problems: An Overview on Existing Verification Procedures

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Abstract

In this paper we give an overview on verification procedures for the solution of complementarity problems.

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1 Introduction

Complementarity problems are finding more and more attention in applications as well as from a mathematical point of view. Especially numerical methods for solving these problems are of great interest. As it is usually the case with numerical methods the computed result delivers an approximation and it has to be confirmed somehow that a solution of the given problem really exists close to the output of the computer. This confirmation is done by the computer and is called verification.

In this survey paper we are mainly concerned with this last aspect. After formulating the complementarity problem and mentioning some well-known applications, we show by a simple example, that verification is a must. We continue by introducing a general verification procedure for complementarity problems. After that we show that for a special class of problems there exists a simple verification procedure. Finally we consider problems with interval data which from a practical point of view always have to be considered if one is taking into account rounding errors, for example.

2 Complementarity Problems and Applications

Given a (nonlinear) mapping $l : \mathbb{R}^n_+ \to \mathbb{R}^n$, where $\mathbb{R}^n_+$ denotes the set of vectors with nonnegative components, the problem

"Find $z \in \mathbb{R}^n$ such that

$$\left.\begin{aligned} z &\geq 0 \\ l(z) &\geq 0 \\ z^T l(z) &= 0 \end{aligned}\right\} \tag{2.1}$$

(or to show that no such z exists)" is called (nonlinear) complementarity problem (NCP).

Equivalent formulations are:

"Find $w, z \in \mathbb{R}$" such that

$$\left.\begin{aligned} w \geq 0, z &\geq 0 \\ w &= l(z) \\ z^T w &= 0 \end{aligned}\right\} \tag{2.2}$$

(or to show that no such w, z exist)",

or:

"Find $z \geq 0$ such that

$$g(z) = \min\,(z, l(z)) = 0\,, \tag{2.3}$$

where the minimum is taken componentwite (or to show that no such z exists)".

If

$$l(z) = Mz + q,$$

where $M \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$ is a given matrix and a given vector, respectively, then the problem is called linear (LCP).

Many problems in science, engineering and economics either arise naturally or can be reformulated as a complementarity problem. We consider a problem from optimization.

Example 1 (Quadratic programming (QP))

"Minimize

$$f(x) = c^T x + \frac{1}{2}\, x^T Q x$$

subject to

$$Ax \geq b$$

$$x \geq 0\text{''},$$

where $Q \in \mathbb{R}^{n \times n}$ is symmetric, $c \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. If $Q = O$, then we have a so-called linear programming problem.

It is well-known that if $x \in \mathbb{R}^n$ is a locally optimal solution, then there exists a $y \in \mathbb{R}^n$ such that $(x, y)^T$ satisfies the Karush-Kuhn-Tucker conditions

$$\left.\begin{array}{ll} u = c + Qx - A^T y \geq 0 \,, & x \geq 0 \,,\ x^T u = 0 \\ v = -\, b + Ax \geq 0 & , \ y \geq 0 \,,\ y^T v = 0 \end{array}\right\} . \tag{2.4}$$

Defining the block matrix

$$M = \begin{pmatrix} Q & A^T \\ A & O \end{pmatrix}$$

and the block vectors

$$q = \begin{pmatrix} c \\ -\, b \end{pmatrix} \,,\ z = \begin{pmatrix} x \\ y \end{pmatrix},$$

respectively, (2.4) can be written as

$$\left.\begin{array}{l} z \geq 0 \\ Mz + q \geq 0 \\ z^T (Mz + q) = 0 \end{array}\right\},$$

which is an (LCP).

If Q is not only symmetric but also positive semi-definite (i.e., $f(x)$ is convex), then (2.4) is not only necessary but also sufficient for x to be a globally optimal solution of the (QP).

A series of further problems which lead to or which can be formulated as a complementarity problem, are the following: contact problem, porous flow problem, obstacle problem, journal bearing problem, elastic plastic torsion problem. For details see [7].

3 Verification of Solutions of Complementarity Problems

We start with a simple example which shows that verification of solutions is a must if one has computed an approximation.

Example 2 (see [1])

Let $\quad l(z) = Mz + q$ where

$$M = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix} \text{ and } q = \begin{pmatrix} 2 \\ 1 \\ -\, 10^{-6} \end{pmatrix} .$$

We consider the formulation (2.2) of the complementarity problem.

For the "approximate solution"

$$z = \begin{pmatrix} 10^{-6} \\ 10^{-6} \\ 1 \end{pmatrix}, \quad w = \begin{pmatrix} 3 \\ 2 \\ 10^{-6} \end{pmatrix}, \tag{3.1}$$

one obtains

$$\|\, l(z) \;-\; w \,\|_\infty = \|\, Mz + q - w \,\|_\infty = 4 \cdot 10^{-6}$$

and

$$z^T w = 6 \cdot 10^{-6}.$$

In many iterative methods (e.g. for interior-point-methods) the condition

$$\max\,\{z^T w\;,\; \|\, Mz + q \;-\; w \,\|_\infty\} \le \epsilon$$

is used as a stopping criteria for some fixed given ϵ. A pair $(z, w)^T$ which fulfils this inequality is then called an "ϵ-approximate solution". In this sense the given vectors z, w form a $6 \cdot 10^{-6}$-approximate solution. However, it can be shown that there is no exact solution of the (LCP) within an $\|\cdot\|_\infty$ distance of 0.25 from this ϵ-approximate solution with $\epsilon = 6 \cdot 10^{-6}$. We will come back to this statement later, again.

Our starting point for the verification of solutions is the equivalent formulation (2.3) of a complementarity problem.

However, we start by explaining the general idea, which is independent of the underlying equation (2.3):

Assume that we have given a continuous mapping

$$H : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$$

and an interval vector $[x] \subseteq D$. A mapping

$$\delta H : [x] \times [x] \to \mathbb{R}^{n \times n}$$

is called slope, if

(A) $\qquad H(x) - H(y) = \delta H(x, y)(x - y)\;,\; x, y \in [x]$

holds. Assume that there exists an interval matrix $\delta H(x, [x])$ such that

(B) $\qquad \delta H(x, y) \in \delta H(x, [x])$

for all $y \in [x]$ and some fixed $x \in [x]$.

Define for a nonsingular matrix $A \in \mathbb{R}^{n\times n}$ the interval vector

$$L(x, A, [x]) := x - A^{-1}H(x) + (I - A^{-1}\delta H(x, [x]))([x] - x).$$

Then, the following hold:

a) If $L(x, A, [x]) \subseteq [x]$, then there exists an $x^* \in [x]$ such that $H(x^*) = 0$.

b) If $L(x, A, [x]) \cap [x] = \emptyset$ (empty set), then $H(x) \neq 0$ for all $x \in [x]$.

In Moore [9] these results were obviously the first time applied to verify the existence of a zero x^* of a mapping H in a given interval vector $[x]$. A proof of the statements a) and b), can also be found in [2]. We are now going to apply these statements to the problem (2.3), that is we replace the general case $H(x) = 0$
by

$$g(z) = \min(z, l(z)) = 0,$$

where $l(z)$ is given. First of all we have to find a slope $\delta g(x, y)$ for which

$$\text{(A)} \qquad g(x) - g(y) = \delta g(x, y)$$

for all $x, y \in [x]$ holds, where $[x]$ is some given interval vector. Forthermore, we have to bound the slope by an interval matrix $\delta g(x, [x])$:

$$\text{(B)} \qquad \delta g(x, y) \in \delta g(x, [x])$$

for some fixed $x \in [x]$ and all $y \in [x]$.

We first consider (A): Define for each $i \in \{1, 2, \dots, n\}$ the sets

$$S_i^+ = \{x \in [x] |\ l_i(x) > x_i\},$$
$$S_i^- = \{x \in [x] |\ l_i(x) < x_i\},$$
$$S_i^0 = \{x \in [x] |\ l_i(x) = x_i\},$$

where $l(x) = (l_i(x))$. Then

$$g_i(x) = \begin{cases} x_i, & x \in S_i^+ \\ l_i(x), & x \in S_i^- \\ l_i(x) = x_i, & x \in S_i^0. \end{cases}$$

Using this representation of $g_i(x)$ we obtain the nine cases from table 1 for the *i-th* row $\delta g_i(x, y)$ of the slope matrix $\delta g(x, y)$, where e_i denotes the *i-th* unit vector.

Table 1: $\delta g_i(x,y)$

$x \setminus y$	S_i^+	S_i^-	S_i^0
S_i^+	e_i^T	$\alpha_i(\delta l_i(x,y) - e_i^T) + e_i^T$	e_i^T
S_i^-	$\beta_i(\delta l_i(x,y) - e_i^T) + e_i^T$	$\delta l_i(x,y)$	$\delta l_i(x,y)$
S_i^0	e_i^T	$\delta l_i(x,y)$	e_i^T

Furthermore, $\delta l_i(x,y)$ denotes a slope (vector) of the *i-th* component of the mapping $l(x)$:

$$l_i(x) - l_i(y) = \delta l_i(x,y)(x-y),$$

and

$$\alpha_i = \frac{y_i - l_i(y)}{(\delta l_i(x,y) - e_i^T)(x-y)},$$

$$\beta_i = \frac{l_i(x) - x_i}{\delta l_i(x,y) - e_i^T)(x-y)}.$$

We prove this for the case $x \in S_i^+, y \in S_i^-$, for example. The proof for the remaining cases can be performed similarly. If $x \in S_i^+, y \in S_i^-$, then

$$\begin{aligned} g_i(x) - g_i(y) &= x_i - l_i(y) \\ &= y_i - l_i(y) + e_i^T(x-y) \\ &= \frac{y_i - l_i(y)(\delta l_i(x,y) - e_i^T)(x-y)}{(\delta l_i(x,y) - e_i^T)(x-y)} + e_i^T(x-y) \\ &= (\alpha_i(\delta l_i(x,y) - e_i^T) + e_i^T)(x-y) \\ &= \delta g_i(x,y)(x-y). \end{aligned}$$

Concerning (B), let $x \in [x]$ be fixed and consider the nonlinear programming problems

$$\min_{y \in [x]} \{y_i - l_i(y)\}$$

and

$$\max_{y\in[x]}\{y_i - l_i(y)\}.$$

Let $y^{i,1}$ and $y^{1,2}$, respectively, be solutions of these problems, then if

$$\delta g_i(x,[x]) = \begin{cases} e_i^T\ , & y^{i,2} \in S_i^+ \cup S_i^0 \\ \delta l_i(x,[x])\ , & y^{i,1} \in S_i^- \cup S_i^0 \\ [0,\alpha_i](\delta l_i(x,[x]) - e_i^T) + e_i^T, & x \in S_i^+ \cup S_i^0, \\ & y^{i,2} \in S_i^- \\ [\beta_i, 1](\delta l_i(x,[x]) - e_i^T) + e_i^T, & x \in S_i^-, \\ & y^{i,1} \in S_i^+ \end{cases}$$

(B) holds. For details see [1].

Exploiting the preceding ideas we get the following algorithm for the verification of a solution:

Algorithm:

Let $r > 0$ be a given tolerance and let x be an approximate solution of

$$g\ (z) = \min(z, l(z)) = 0.$$

Define the interval vector

$$[x] = (x + r\ [-e, e]) \cap \mathbb{R}_+$$

where $e = (1, 1, \ldots, 1)^T$.

Choose a nonsingular A and compute $L(x, A, [x])$.

If $L(x, A, [x]) \subseteq [x]$ then $g(z^*) = 0$ for some $z^* \in [x]$.

If $L(x, A, [x]) \cap [x] = \emptyset$(empty set), then $g(z) \neq 0$ for all $z \in [x]$.

In [1] this algorithm was applied in order to prove the conclusion from example 1. In [3] there are also numerical results for the nonlinear case.

4 Special Complementarity Problems

In certain cases it is possible to avoid the laborous bounding of the slope given by the elements of table 1. Starting again with the formulation (2.3), we easily see that a nonnegative solution of (2.3) is a fixed point of the mapping

$$p\ :\ \mathbb{R}^n_+ \to \mathbb{R}^n_+$$

where

$$p(x) = \max\{0, x - D \cdot l(x)\}$$

and where $D = \text{diag}\ (d_i)$ is a diagonal matrix with positive elements in the main diagonal. The maximum is again taken componentwise. Please note that the preceding statement is not true in general for a nonsingular, nonnegative matrix, which is not a diagonal matrix.

Assume now that l' has a so-called interval arithmetic evaluation $l'([x])$, that is $l'(x) \in l'([x])$ holds for all $x \in [x]$ and an interval vector [x]. (Such an $l'([x])$ exists and can be computed if the Jacobian of l exists and can be evaluated for the given interval vector. For details see any book on interval arithmetic.)

Let now [x] be given. Let $l'([x])$ denote the interval arithmetic evaluation of l' over $[x]$. If

$$\Gamma(x, [x], D) := \max\{0, x - D \cdot l[x] + (I - Dl'([x]))([x] - x)\}$$

is contained in $[x], \Gamma(x, [x], D) \subseteq [x]$, where $x \in [x]$ is fixed, then there exists a solution z^* of (2.3) in $\Gamma(x, [x], D)$ (and therefore also in $[x]$). All solutions of (2.3) contained in $[x]$ are also contained in $\Gamma(x, [x], D)$. From the last statement it follows that there is no solution of (2.3) in $[x]$ if $\Gamma(x, [x], D) \cap [x] = \emptyset$. (In the definition of $\Gamma(x, [x], D)$, there appears an interval vector $[x]$ and the maximum of the zero vector 0 and an interval vector, say $[y]$, has to be performed. This maximum is defined as follows: Let the interval vector $[y]$ have the lower bound $\underline{y}$ and the upper bound $\overline{y}$, respectively. Then $\max\{0, [y]\}$ is the interval vector with lower bound $\max\{0, \underline{y}\}$ and upper bound $\max\{0, \overline{y}\}$, respectively. The maximum of two real vectors is formed componentwise.)

If the problem is linear,

$$l(z) = Mz + q$$

with a given H-matrix M with positive diagonal elements, it is easy to find an interval vector $[x]$ and a diagonal matrix with positive elements in the main diagonal such that $\Gamma(x, [x], D) \subseteq [x]$. Furthermore we have a simple iterative method which computes a sequence of interval vectors, all containing the (in this case for arbitrary q) unique solution z^*, and converging to z^*.

We first repeat some properties of H-matrices. Then we consider the iterative method and finally we construct an interval vector containing z^*. Define for the given matrix $M = (m_{ij})$ the so-called comparison

matrix $\tilde{M} = (\tilde{m}_{ij})$
by

$$\tilde{m}_{ij} = \begin{cases} |m_{ii}| \text{ if } i = j \\ -|m_{ij}| \text{ if } i \neq j \end{cases}.$$

M is called an H-matrix iff there exists a positive vector d such that $\tilde{M}d > 0$.

The diagonal elements of an H-matrix are different from zero. Therefore, if M is real, they are either positive or negative. If the diagonal elements of a (real) H-matrix are all positive, then this matrix belongs to the set of so-called P-matrices (see [7]). If M is a P-matrix then the (LCP) has a unique solution for each q.

We now assume that M is an H-matrix with positive diagonal elements $m_{ii} > 0, i = 1, 2, \ldots, n$. Since $l'(x) = M$ in this case, we obtain

$$\Gamma(x, [x], D) = \max\{0, x - D(Mx + q) + (I - DM)([x] - x)\}.$$

Assume now that

$$\Gamma(x, [x], D) \subseteq [x]$$

for some given $[x]$. Define $[x^0] := [x]$ and consider the iterative method

$$[x^{k+1}] = \Gamma(x^k, [x^k], D) \cap [x^k] \tag{4.1}$$

where

$$x^k = m([x^k])$$

is the center of $[x^k]$.

Then the following hold:

a) If the (unique) solution z^* of (2.3) is contained in $[x^0]$ then the iteration method (4.1) is well-defined and $\lim_{k\to\infty} [x^k] = z^*$.

b) If $z^* \notin [x^0]$ then the intersection becomes empty after a finite number of steps.

For details see [5], where also the method for finding $[x^0]$ with $z^* \in [x^0]$ is discussed. In that paper also a couple of numerical examples are presented.

5 Interval Data

Representing the given data of a given complementarity problem on a computer usually implies rounding of the data. Hence on the computer a problem different from the given one has to be solved. One can overcome this difficulty by enclosing the data by intervals on the computer and by considering a whole set of complementarity problems. Then the problem consists of computing the set of the solutions of all problems.

Another interesting case, in which interval data have to be considered, was given by Schäfer [10,11]. He showed that discretizing a certain free boundary problem leads in a natural manner to a complementarity problem with interval data if the discretization error is taken into account.

After the formulation of the problem we consider some iterative methods for the inclusion of the solution set and show convergence of these methods for a certain class of problems.

Let there be given an interval matrix $[M] = ([m_{ij}]) \in I\!R^{n\times n}$ and an interval vector $[q] \in ([q_i]) \in I\!R^n$. Then we consider the set $\sum$ of the solutions of all possible linear complementarity problems:

$$\sum := \{z \in \mathbb{R}^n \mid z \geq 0\,,\ q{+}Mz \geq 0\,,\ z^T(q{+}Mz) = 0\,,\ M \in [M]\,,\ q \in [q]\}.$$

The problem consists of finding an (as small as possible) interval vector $[x] \in I\!R^n$, such that $\sum \subseteq [x]$.

We now consider the case, that the given interval matrix is a so-called (interval) H-matrix. For the definition of an H-matrix in the interval case we generalize first the definition of the comparison matrix of a given matrix (see section 3).

Given the interval matrix $[M] = ([m_{ij}])$, where $[m_{ij}] = [\underline{m}_{ij}, \overline{m}_{ij}]$, the comparison matrix $\tilde{M} \in \mathbb{R}^{n\times n}$ is a real matrix with

$$\tilde{m}_{ij} = \begin{cases} min\{|m_{ij}| \mid m_{ij} \in [m_{ij}]\}, & i = j \\ -\,|[m_{ij}]|, & i \neq j, \end{cases}$$

where the absolute value of the interval $[m_{ij}]$ is a real number defined by $|\,[m_{ij}]\,| = \max\,\{|\,\underline{m}_{ij}\,|\,,\,|\,\overline{m}_{ij}\,|\}$. The interval matrix $[M]$ is called interval H-matrix if the comparison matrix is an H-matrix, that is, if there exists a positive vector, such that $\tilde{M}d > 0$.

Now assume that the given interval matrix $[M] = ([m_{ij}])$ is an interval H-matrix with $\underline{m}_{ii} > 0, i = 1, 2, \ldots, n$. We split $[M]$ into its diagonal part $[D]$ and its off-diagonal part $-[R] : [M] = [D] - [R]$. Define the diagonal matrix $[\hat{D}]$ by

$$[\hat{D}] \ := \operatorname{diag}\,(\frac{1}{[m_{ii}]}).$$

Then we consider the iteration method

$$[x^0] = [\underline{x}^0, \overline{x}^0] \in I\!\!R^n, \underline{x}^0 \geq 0$$

$$[x^{k+1}] = \max\{0\ ,\ [\hat{D}]([R]\ [x^k] - [q]\}\ ,\ k = 0, 1, 2, \dots\ . \qquad \text{(T)}$$

(T) has the following properties:

a) $\lim_{k\to\infty} [x^k] = [x^*]$ with

$[x^*] = \max\{0\ ,\ [\hat{D}]\ ([R]\ [x^*]\ -\ [q])\}$;

$[x^*]$ is unique.

b) $\sum \subseteq [x^*]$.

The proof of a) follows by application of the Banach fixed point theorem to the mapping

$$g([x]) = \max\{0, [\hat{D}]\ ([R]\ [x]\ -\ [q])\}$$

using the fact that the spectral radius of the real matrix $|\,[\hat{D}]||\,[R]|$ is less than one. The second part b) follows by simple interval arithmetic manipulations.

The introduced method (T) can be considered as a straight forward generalization of the well-known total step method to complementarity problems (with interval data). Another choice would be to consider the Gauss-Seidel-method (GS) and its generalization. It has been shown in [4] that under the conditions assumed for (T), (GS) is also convergent (to the same limit $[x^*]$ as (T)). Introducing a so-called relaxation parameter ω in order to speed up (GS) by the successive over relaxation method (SOR) is not advantageous if interval data are considered since in this case the fixed point $[x^*]$ of (T) may be inflated for certain values of ω. For details see [4].

It is interesting to raise the question whether the limit $[x^*]$ of (T) is the smallest interval vector (with respect to inclusion) for which $\sum \subseteq [x^*]$ holds. In general this is not the case under our assumption that $[M]$ is an interval matrix with $\underline{m}_{ii} > 0\ ,\ i = 1, 2, \dots, n$. However, by restricting the set of matrices further, this is actually true: If we assume that for the given interval H-matrix not only $\underline{m}_{ii} > 0$ but also $\overline{m}_{ij} \leq 0\ ,\ i \neq j$ (such a matrix is called an interval M(inkowski)-matrix), then $[x^*]$ is optimal in the sense that there exists no smaller interval vector with respect to inclusion such that b) holds.

For a proof see [4], where also a couple of numerical examples with interval data can be found.

6 Newton's Method

In this section we apply the idea of Newton's method to the (NCP) from (2.1).

Define the linearized mapping l^k by

$$l^k(z) = l(z^k) + l'(z^k)(z - z^k).$$

Then we consider the following algorithm, which may be considered as a generalization of Newton's method for the solution of nonlinear equations:

$$(NM) \begin{cases} \text{Choose } z^0; \\ \text{For } k = 0, 1, \ldots \text{ compute the solution } z^{k+1} \text{ of the (LCP)} \\ z \geq 0 \ , \ l^k(z) \geq 0 \ , \ z^T l^k(z) = 0. \end{cases}$$

Concerning the existence and convergence of the sequence $\{z^k\}_{k=0}^{\infty}$ the following result has been proven by Z. Wang [12]:

Let $D \subseteq \mathbb{R}^n$ be open and $\mathbb{R}^n_+ \subseteq D$. Assume that $l : D \longrightarrow \mathbb{R}^n$ is differentiable, $D_0 \subseteq D$ is convex and that

$$||l'(x) - l'(y)||_\infty \leq \gamma ||x - y||_\infty, x, y \in D_0.$$

Suppose there exists a starting point $z^0 \in D_0$ such that $l'(z^0)$ is an H-matrix with positive diagonal elements and

$$||\overline{l'(z^0)}^{-1}||_\infty \leq \beta,$$

where the bar denotes the comparison matrix. Denote by z^1 the first iterate of (NM) starting with z^0. Let $||z^1 - z^0||_\infty \leq \eta$. If $h = \beta\gamma\eta \leq \frac{1}{2}$ and $\overline{S}(z^0, r^*) \subseteq D_0$, where $r^* = (1 - \sqrt{1 - 2h})/\beta\gamma$, then the sequence $\{z^k\}$ is well defined, remains in $\overline{S}(z^0, r^*)$ and converges to a solution z^* of (NCP) which exists in the ball $\overline{S}(z^0, r^*)$. z^* is unique in the ball $S(z^0, r^{**})$, where $r^{**} = (1 + \sqrt{1 - 2h})/\beta\gamma$. The error estimation

$$||z^k - z^*||_\infty \leq \frac{1}{2^{k-1}} (2h)^{2^k - 1} \eta = (\beta\gamma 2^k)^{-1} (2h)^{2^k}$$

holds. The preceding statements also hold for the 1-norm.

If $l'(x)$ is positive definite, let $\widetilde{l'(x)} = \frac{1}{2}(l'(x) + (l'(x))^T)$. If $\overline{l'(x^0)}$ is replaced by $\widetilde{l'(x^0)}$, the preceding statements also hold in the 2-norm.

Using the error estimation for $z^k - z^*$, the verification of a solution in a ball can be verified.

7 Further Results

In the paper [8], Mathias and Pang have shown that for the unique solution z^* of the (linear) complementarity problem (2.1) with $l(z) = Mz + q$, where M is a so-called P-matrix, the estimation

$$||z - z^*||_\infty \leq \frac{1 + ||M||_\infty}{c(M)} ||r(z)||_\infty$$

with

$$r(z) = \min\{z, Mz + q\},$$

$$c(M) = \min_{||z||_\infty = 1} \{\max_{1 \leq i \leq n} z_i(Mz_i)_i\}$$

holds for all $z \in \mathbb{R}^n$. However, $c(M)$ is not easy to find for a given approximation z of z^*. Therefore the following estimation by X. Chen and S. Xiang [6] is of great importance: For each $p \geq 1$ it holds for all $z \in \mathbb{R}^n$

$$||z - z^*||_p \leq \max_{D \in [0,1]^n} ||(I - D + DM)^{-1}||_p ||r(z)||_p,$$

where $D = \text{diag}\,(d_1, \ldots, d_n)$. Furthermore

$$\max_{D \in [0,1]^n} ||(I - D + DM)^{-1}||_\infty$$

$$\leq \frac{\max\{1, ||M||_\infty\}}{c(M)} = \frac{1 + ||M||_\infty}{c(M)} - \frac{\min\{1, ||M||_\infty\}}{c(M)}.$$

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References

[1] G. E. Alefeld, X. Chen, F. A. Potra: Numerical validation of solutions of linear complementarity problems. Numer. Math. 83, 1-23 (1999).

[2] G E. Alefeld, G. Mayer: Interval Analysis: Theory and Applications. Journal of Comp. and Appl. Math. 121, 421-464 (2000).

[3] G. E. Alefeld, X. Chen, F. A. Potra: Numerical Validation of solutions of complementarity problems: The nonlinear case. Numer. Math 92, 1-16 (2002).

[4] G. E. Alefeld, U. Schäfer: Iterative Methods for Linear Complementarity Problems with Interval Data. Computing 70, 25-259 (2003).

[5] G. E. Alefeld, Z. Wang, Z. Shen: Enclosing solutions of linear complementarity problems for H-matrices. Reliable Computing 10, 423-435 (2004).

[6] X. Chen, S. Xiang: Computation of error bounds for P-matrix linear complementarity problems. Math. Programming, 106, 513-525 (2006).

[7] R. W. Cottle, J.-S. Pang, R. E. Stone: The linear complementarity problem. Academic Press, 1992.

[8] R. Mathias, J. S. Prang: Error bounds for the linear complementarity problem with a P-matrix. Linear Algebra Appl., 132, 123-136 (1990).

[9] R. E. Moore: A Test for Existence of Solutions to Nonlinear Systems. SIAM J. Numer. Anal. 14, 611-615 (1977).

[10] U. Schäfer: Das lineare Komplementaritätsproblem mit Intervalleinträgen. Dissertation. University of Karlsruhe (1999).

[11] U. Schäfer: An enclosure method for free boundary problems based on a linear complementarity problem with interval data. Numer. Funct. Anal. Optim. 22, 991-1011 (2001).

[12] Z. Wang: Semilocal Convergence of Newton's Method for Finite-Dimensional Variational Inequalities and Nonlinear Complementarity Problems. PhD-Thesis. Fakultät für Mathematik. University of Karlsruhe (2005).