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A splitting theorem for local cohomology and its applications ☆

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ABSTRACT

Let R be a commutative Noetherian ring and M a finitely generated R-module. We show in this paper that, for an integer t, if the local cohomology module $H^i_{\mathfrak{a}}(M)$ with respect to an ideal $\mathfrak{a}$ is finitely generated for all $i < t$, then

$$H^i_{\mathfrak{a}}(M/xM) \cong H^i_{\mathfrak{a}}(M) \oplus H^{i+1}_{\mathfrak{a}}(M)$$

for all $\mathfrak{a}$-filter regular elements x contained in a enough large power of $\mathfrak{a}$ and all $i < t - 1$. As consequences we obtain generalizations, by very short proofs, of the main results of M. Brodmann and A.L. Faghani [M. Brodmann, A.L. Faghani, A finiteness result for associated primes of local cohomology modules, Proc. Amer. Math. Soc. 128 (2000) 2851–2853] and of H.L. Truong and the first author [N.T. Cuong, H.L. Truong, Asymptotic behavior of parameter ideals in generalized Cohen–Macaulay module, J. Algebra 320 (2008) 158–168].

1. Introduction

A finitely generated module M of dimension $d > 0$ over a Noetherian local ring $(R, \mathfrak{m})$ is called a generalized Cohen–Macaulay module (see [3]), if there exists a positive integer k such that $\mathfrak{m}^k H^i_{\mathfrak{m}}(M) = 0$ for all $i < d$, where $H^i_{\mathfrak{m}}(M)$ is the i-th local cohomology module of M with respect to the maximal ideal $\mathfrak{m}$. Then the following split property of local cohomology modules is useful in

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the theory of generalized Cohen–Macaulay modules (see [11]): For a parameter element x of M there exists a enough large integer n such that $H^i_{\mathfrak{m}}(M/x^nM) \cong H^i_{\mathfrak{m}}(M) \oplus H^{i+1}_{\mathfrak{m}}(M)$ for all $i < d-1$. It should be noted here that this integer n is in general dependent on the choice of the element x. It raises to the following natural question.

Question. Let M be a generalized Cohen–Macaulay module. Does there exist a positive integer n such that for any parameter element x of M contained in $\mathfrak{m}^n$, it holds $H^i_{\mathfrak{m}}(M/xM) \cong H^i_{\mathfrak{m}}(M) \oplus H^{i+1}_{\mathfrak{m}}(M)$ for all $i < d-1$?

The purpose of this paper is not only to find an answer to this question but also to prove a more general split property of local cohomology modules as follows. Let R be a Noetherian ring (R is not necessary to be a local ring) and $\mathfrak{a}$ an ideal of R. An element $x \in \mathfrak{a}$ is called an $\mathfrak{a}$-filter regular element of M if $x \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass} M \setminus V(\mathfrak{a})$, where $V(\mathfrak{a})$ is the set of all prime ideals of R containing $\mathfrak{a}$.

Theorem 1.1. *Let M be a finitely generated module over a Noetherian ring R and $\mathfrak{a}$ an ideal of R. Let t and n_0 be positive integers such that $\mathfrak{a}^{n_0} H^i_{\mathfrak{a}}(M) = 0$ for all $i < t$. Then, for all $\mathfrak{a}$-filter regular element $x \in \mathfrak{a}^{2n_0}$ of M, it holds*

$$H^i_{\mathfrak{a}}(M/xM) \cong H^i_{\mathfrak{a}}(M) \oplus H^{i+1}_{\mathfrak{a}}(M)$$

for all $i < t-1$, and

$$0 :_{H^{t-1}_{\mathfrak{a}}(M/xM)} \mathfrak{a}^{n_0} \cong H^{t-1}_{\mathfrak{a}}(M) \oplus 0 :_{H^t_{\mathfrak{a}}(M)} \mathfrak{a}^{n_0}.$$

It is well known that every parameter element of M is an $\mathfrak{m}$-filter regular element, if M is a generalized Cohen–Macaulay module. Therefore Theorem 1.1 gives a complete affirmative answer for the question above, where the integer n is just $n = \min\{k \mid \mathfrak{m}^k H^i_{\mathfrak{m}}(M) = 0,\ i = 0, \dots, d-1\}$. The key point for proving Theorem 1.1 is as follows. Let x and t be as in Theorem 1.1. From the short exact sequence $0 \to M/H^0_{\mathfrak{a}}(M) \xrightarrow{x} M \to M/xM \to 0$ we obtain short exact sequences

$$0 \longrightarrow H^i_{\mathfrak{a}}(M) \longrightarrow H^i_{\mathfrak{a}}(M/xM) \longrightarrow H^{i+1}_{\mathfrak{a}}(M) \longrightarrow 0, \quad i = 0, \dots, t-2. \qquad (*)$$

So for each $i < t-1$ we can consider the short exact sequence $(*)$ as an element of the group of extensions $\operatorname{Ext}^1_R(H^{i+1}_{\mathfrak{a}}(M), H^i_{\mathfrak{a}}(M))$ (see, Chapter 3, [10]). Then, the splitting of sequence $(*)$ is equivalent to say that it is the zero-element of this group. We will give some properties of the (Bear) sum and the R-module structure of this group of extensions in the next section. The proof of Theorem 1.1 will be done in Section 3. The last section is involved to find applications of Theorem 1.1. Especially, we show that the main theorems of M. Brodmann and A.L. Faghani [2], and of H.L. Truong and the first author [4] are immediate consequences of Theorem 1.1.

2. The extension module Ext^1

In this section, let $\mathfrak{a}$ be an ideal of a Noetherian ring R, and M a finitely generated R-module. It is well known that for a positive integer t, $H^i_{\mathfrak{a}}(M)$ is finitely generated for all $i < t$ iff there exists a positive integer n_0 such that $\mathfrak{a}^{n_0} H^i_{\mathfrak{a}}(M) = 0$ for all $i < t$. An element $x \in \mathfrak{a}$ is called an $\mathfrak{a}$-filter regular element of M if $x \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass} M \setminus V(\mathfrak{a})$, where $V(\mathfrak{a})$ is the set of all prime ideals of R containing $\mathfrak{a}$. It should be noted that there always exist $\mathfrak{a}$-filter regular elements. Moreover, if $x \in \mathfrak{a}^{n_0}$ is an $\mathfrak{a}$-filter regular element of M, then the short exact sequence

$$0 \longrightarrow M' \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0,$$

where $M' = M/H^0_{\mathfrak{a}}(M)$, reduces short exact sequences

$$0 \longrightarrow H^i_{\mathfrak{a}}(M) \longrightarrow H^i_{\mathfrak{a}}(M/xM) \longrightarrow H^{i+1}_{\mathfrak{a}}\big(M'\big) \longrightarrow 0,$$

for all $i < t-1$. This situation is a special case of the following consideration: given an integer t, an ideal $\mathfrak{a}$ of R and a submodule U of M. Set $\overline{M} = M/U$. We say that an element $x \in \mathfrak{a}$ satisfies the condition ($\sharp$) if $0:_M x = U$, and the short exact sequence

$$0 \longrightarrow \overline{M} \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0$$

reduces short exact sequences

$$0 \longrightarrow H^i_{\mathfrak{a}}(M) \longrightarrow H^i_{\mathfrak{a}}(M/xM) \longrightarrow H^{i+1}_{\mathfrak{a}}(\overline{M}) \longrightarrow 0$$

for all $i < t-1$.

Proposition 2.1. *Let $M, U, \overline{M}, \mathfrak{a}$ and t be as above. Suppose that x, y are elements in $\mathfrak{a}$ such that x and xy satisfy the condition ($\sharp$), and $yH^i_{\mathfrak{a}}(M) = 0$ for all $i < t$. Then, for all $i < t-1$, we have*

$$H^i_{\mathfrak{a}}(M/xyM) \cong H^i_{\mathfrak{a}}(M) \oplus H^{i+1}_{\mathfrak{a}}(\overline{M}).$$

Moreover, if $H^t_{\mathfrak{a}}(\overline{M}) \cong H^t_{\mathfrak{a}}(M)$, we have

$$0:_{H^{t-1}_{\mathfrak{a}}(M/xyM)} x \cong 0:_{H^{t-1}_{\mathfrak{a}}(M)} x \oplus 0:_{H^t_{\mathfrak{a}}(\overline{M})} x.$$

Proof. Since $U = 0:_M x = 0:_M xy$, we have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \overline{M} & \xrightarrow{x} & M & \xrightarrow{p_1} & M/xM & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & \overline{M} & \xrightarrow{xy} & M & \xrightarrow{p_2} & M/xyM & \longrightarrow & 0
\end{array}$$

with exact rows, p_1, p_2 are natural projections, and f is the induced homomorphism. We get by applying the functor $H^i_{\mathfrak{a}}(\bullet)$ to the above diagram for all $i < t-1$ the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^i_{\mathfrak{a}}(M) & \xrightarrow{H^i_{\mathfrak{a}}(p_1)} & H^i_{\mathfrak{a}}(M/xM) & \xrightarrow{\delta^i_1} & H^{i+1}_{\mathfrak{a}}(\overline{M}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle H^i_{\mathfrak{a}}(f)} & & \downarrow{\scriptstyle id} & & \\
0 & \longrightarrow & H^i_{\mathfrak{a}}(M) & \xrightarrow{H^i_{\mathfrak{a}}(p_2)} & H^i_{\mathfrak{a}}(M/xyM) & \xrightarrow{\delta^i_2} & H^{i+1}_{\mathfrak{a}}(\overline{M}) & \longrightarrow & 0,
\end{array}$$

where δ^i_1, δ^i_2 are connected homomorphisms. Moreover, since $yH^i_{\mathfrak{a}}(M) = 0$ for all $i < t-1$, $H^i_{\mathfrak{a}}(f) \circ H^i_{\mathfrak{a}}(p_1) = 0$. Therefore there exists a homomorphism

$$\epsilon^i : H^{i+1}_{\mathfrak{a}}(\overline{M}) \cong \operatorname{coker} H^i_{\mathfrak{a}}(p_1) \longrightarrow H^i_{\mathfrak{a}}(M/xyM)$$

for all $i < t-1$, which makes the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^i_{\mathfrak{a}}(M) & \xrightarrow{H^i_{\mathfrak{a}}(p_1)} & H^i_{\mathfrak{a}}(M/xM) & \xrightarrow{\delta^i_1} & H^{i+1}_{\mathfrak{a}}(\overline{M}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle H^i_{\mathfrak{a}}(f)} & \swarrow{\scriptstyle \epsilon^i} & \downarrow{\scriptstyle id} & & \\
0 & \longrightarrow & H^i_{\mathfrak{a}}(M) & \xrightarrow{H^i_{\mathfrak{a}}(p_2)} & H^i_{\mathfrak{a}}(M/xyM) & \xrightarrow{\delta^i_2} & H^{i+1}_{\mathfrak{a}}(\overline{M}) & \longrightarrow & 0
\end{array}$$

commutative for all $i < t-1$. Hence $\delta^i_2 \circ \epsilon^i = id$, and so we get

$$H^i_{\mathfrak{a}}(M/xyM) \cong H^i_{\mathfrak{a}}(M) \oplus H^{i+1}_{\mathfrak{a}}(\overline{M}),$$

for all $i < t-1$.

In the case $H^t_{\mathfrak{a}}(\overline{M}) \cong H^t_{\mathfrak{a}}(M)$ and $i = t-1$, we have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^{t-1}_{\mathfrak{a}}(M) & \xrightarrow{H^{t-1}_{\mathfrak{a}}(p_1)} & H^{t-1}_{\mathfrak{a}}(M/xM) & \xrightarrow{\delta^{t-1}_1} & 0:_{H^t_{\mathfrak{a}}(\overline{M})} x & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle H^{t-1}_{\mathfrak{a}}(f)} & & \downarrow{\scriptstyle \alpha} & & \\
0 & \longrightarrow & H^{t-1}_{\mathfrak{a}}(M) & \xrightarrow{H^{t-1}_{\mathfrak{a}}(p_2)} & H^{t-1}_{\mathfrak{a}}(M/xyM) & \xrightarrow{\delta^{d-1}_2} & 0:_{H^t_{\mathfrak{a}}(\overline{M})} xy & \longrightarrow & 0,
\end{array}$$

where $\alpha\colon 0:_{H^t_{\mathfrak{a}}(\overline{M})} x \to 0:_{H^t_{\mathfrak{a}}(\overline{M})} xy$ is injective. With similar method as used in the cases $i < t-1$, there exists a homomorphism $\epsilon^{t-1}\colon 0:_{H^t_{\mathfrak{a}}(\overline{M})} x \to H^{t-1}_{\mathfrak{a}}(M/xyM)$ such that $\delta^{t-1}_2 \circ \epsilon^{t-1} = \alpha$. By applying the functor $\mathrm{Hom}_R(R/(x), \bullet)$ to the above diagram we can check that

$$0:_{H^{t-1}_{\mathfrak{a}}(M/xyM)} x \cong 0:_{H^{t-1}_{\mathfrak{a}}(M)} x \oplus 0:_{H^t_{\mathfrak{a}}(\overline{M})} x. \qquad \square$$

If $x \in \mathfrak{a}$ satisfies the condition ($\sharp$), for each $i < t-1$ we can consider

$$0 \longrightarrow H^i_{\mathfrak{a}}(M) \longrightarrow H^i_{\mathfrak{a}}(M/xM) \longrightarrow H^{i+1}_{\mathfrak{a}}(\overline{M}) \longrightarrow 0$$

as an extension of $H^i_{\mathfrak{a}}(M)$ by $H^{i+1}_{\mathfrak{a}}(\overline{M})$, therefore as an element of $\mathrm{Ext}^1_R(H^{i+1}_{\mathfrak{a}}(\overline{M}), H^i_{\mathfrak{a}}(M))$ (see, Chapter 3, [10]). We denote this element by E^i_x. Especially, if $H^t_{\mathfrak{a}}(\overline{M}) \cong H^t_{\mathfrak{a}}(M)$, we have the short exact sequence

$$0 \longrightarrow H^{t-1}_{\mathfrak{a}}(M) \longrightarrow H^{t-1}_{\mathfrak{a}}(M/xM) \longrightarrow 0:_{H^t_{\mathfrak{a}}(\overline{M})} x \longrightarrow 0.$$

Let n_0 be a positive integer such that $x \in \mathfrak{a}^{n_0}$. Suppose that the short exact sequence above derives the following short exact sequence

$$0 \longrightarrow 0:_{H^{t-1}_{\mathfrak{a}}(M)} \mathfrak{a}^{n_0} \longrightarrow 0:_{H^{t-1}_{\mathfrak{a}}(M/xM)} \mathfrak{a}^{n_0} \longrightarrow 0:_{H^t_{\mathfrak{a}}(\overline{M})} \mathfrak{a}^{n_0} \longrightarrow 0.$$

Then we can consider this exact sequence as an element of $\mathrm{Ext}^1_R(0:_{H^t_{\mathfrak{a}}(\overline{M})} \mathfrak{a}^{n_0}, 0:_{H^{t-1}_{\mathfrak{a}}(M)} \mathfrak{a}^{n_0})$, and denote it by $F^{t-1}_{n_0,x}$. It should be noted here that an extension of R-module A by R-module C is split if it is the zero-element of $\mathrm{Ext}^1_R(C, A)$. The following result is important for the proof of Theorem 1.1.

Theorem 2.2. *Let M, U, $\overline{M}$, $\mathfrak{a}$ and t be as above and $x, y \in \mathfrak{a}$. Then the following statements are true.*

(i) *Suppose that $x, y, x+y$ satisfy the condition ($\sharp$), then $E^i_{x+y} = E^i_x + E^i_y$ for all $i < t-1$. Furthermore, if $H^t_\mathfrak{a}(\overline{M}) \cong H^t_\mathfrak{a}(M)$ and $F^{t-1}_{n_0,x}, F^{t-1}_{n_0,y}$ are determined, then $F^{t-1}_{n_0,x+y}$ is also determined, and we have $F^{t-1}_{n_0,x+y} = F^{t-1}_{n_0,x} + F^{t-1}_{n_0,y}$.*

(ii) *Suppose that x, xy satisfy the condition ($\sharp$), then $E^i_{xy} = yE^i_x$ for all $i < t-1$. Moreover, if $H^t_\mathfrak{a}(\overline{M}) \cong H^t_\mathfrak{a}(M)$ and $F^{t-1}_{n_0,x}$ is determined, then $F^{t-1}_{n_0,xy}$ is also determined and $F^{t-1}_{n_0,xy} = yF^{t-1}_{n_0,x}$. Especially, if $yH^i_\mathfrak{a}(M) = 0$, for all $i < t$, then $F^{t-1}_{n_0,xy} = E^i_{xy} = 0$ for all $i < t-1$.*

Proof. (i) We consider the homomorphism $\varphi : M \to M \oplus M$, $\varphi(m) = (xm, ym)$. Because $U = 0 :_M x = 0 :_M y$ so we have short exact sequence

$$0 \longrightarrow \overline{M} \xrightarrow{\overline{\varphi}} M \oplus M \longrightarrow N \longrightarrow 0,$$

where $N = \operatorname{coker}(\overline{\varphi})$. The following diagram is commutative

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \overline{M} & \xrightarrow{\overline{\varphi}} & M \oplus M & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \Delta_{\overline{M}}} & & \downarrow{\scriptstyle id} & & \downarrow & & \\
0 & \longrightarrow & \overline{M} \oplus \overline{M} & \xrightarrow{x \oplus y} & M \oplus M & \longrightarrow & M/xM \oplus M/yM & \longrightarrow & 0,
\end{array}$$

where $\Delta_{\overline{M}} : \overline{M} \to \overline{M} \oplus \overline{M}$, $\Delta(m) = (m, m)$ is a diagonal homomorphism. Note that the derived homomorphism of $\Delta_{\overline{M}}$ is also a diagonal homomorphism, the homomorphism $\Delta_{H^i_\mathfrak{a}(\overline{M})} : H^i_\mathfrak{a}(\overline{M}) \to H^i_\mathfrak{a}(\overline{M}) \oplus H^i_\mathfrak{a}(\overline{M})$ is diagonal for all $i \geqslant 0$. Therefore, we get by applying the functor $H^i_\mathfrak{a}(\bullet)$ to the above diagram the following commutative diagram

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & H^i_\mathfrak{a}(\overline{M}) & \xrightarrow{\varphi^i} & H^i_\mathfrak{a}(M)^2 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \Delta_{H^i_\mathfrak{a}(\overline{M})}} & & \downarrow{\scriptstyle id} & & \\
\cdots & \longrightarrow & H^i_\mathfrak{a}(\overline{M})^2 & \xrightarrow{x \oplus y} & H^i_\mathfrak{a}(M)^2 & \longrightarrow & \cdots,
\end{array}$$

where $A^2 = A \oplus A$ for an R-module A, and φ^i is derived from $\overline{\varphi}$. Since x, y satisfy the condition ($\sharp$), the homomorphism in the bottom row is zero, for all $i < t$, hence $\varphi^i = 0$ for all $i < t$. Therefore, for all $i < t-1$, the following diagram is commutative

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^i_\mathfrak{a}(M)^2 & \longrightarrow & H^i_\mathfrak{a}(N) & \longrightarrow & H^{i+1}_\mathfrak{a}(\overline{M}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle id} & & \downarrow & & \downarrow{\scriptstyle \Delta_{H^{i+1}_\mathfrak{a}(\overline{M})}} & & \\
0 & \longrightarrow & H^i_\mathfrak{a}(M)^2 & \longrightarrow & H^i_\mathfrak{a}(M/xM) \oplus H^i_\mathfrak{a}(M/yM) & \longrightarrow & H^{i+1}_\mathfrak{a}(\overline{M})^2 & \longrightarrow & 0.
\end{array}$$

For all $i < t-1$, the exact sequence in the bottom row is just $E^i_x \oplus E^i_y$. We denote the exact sequence in the top row by E^i, so

$$E^i = \left(E^i_x \oplus E^i_y\right)\Delta_{H^{i+1}_{\mathfrak{m}}(\overline{M})} \tag{1}$$

for all $i < t - 1$.

Moreover, if $H^t_{\mathfrak{a}}(M) \cong H^t_{\mathfrak{a}}(\overline{M})$, we have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^{t-1}_{\mathfrak{a}}(M)^2 & \longrightarrow & H^{t-1}_{\mathfrak{a}}(N) & \longrightarrow & K_{(x,y)} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle id} & & \downarrow & & \downarrow{\scriptstyle \overline{\Delta}} & & \\
0 & \longrightarrow & H^{t-1}_{\mathfrak{a}}(M)^2 & \longrightarrow & H^{t-1}_{\mathfrak{a}}(M/xM) \oplus H^{t-1}_{\mathfrak{a}}(M/yM) & \longrightarrow & K_x \oplus K_y & \longrightarrow & 0,
\end{array}$$

where $K_{(x,y)} = 0 :_{H^t_{\mathfrak{a}}(\overline{M})} (x, y)$, $K_x = 0 :_{H^t_{\mathfrak{a}}(\overline{M})} x$, $K_y = 0 :_{H^t_{\mathfrak{a}}(\overline{M})} y$, and $\overline{\Delta} : K_{(x,y)} \to K_x \oplus K_y$ defined by $\overline{\Delta}(c) = (c, c)$. Since

$$\operatorname{Hom}_R\left(R/\mathfrak{a}^{n_0}, K_x\right) \cong \operatorname{Hom}_R\left(R/\mathfrak{a}^{n_0}, K_y\right) \cong \operatorname{Hom}_R\left(R/\mathfrak{a}^{n_0}, K_{(x,y)}\right) \cong 0 :_{H^t_{\mathfrak{a}}(\overline{M})} \mathfrak{a}^{n_0},$$

by applying the functor $\operatorname{Ext}^i_R(R/\mathfrak{a}^{n_0}, \bullet)$ to the above diagram we obtain the following commutative diagram

$$\begin{array}{ccc}
0 :_{H^t_{\mathfrak{a}}(\overline{M})} \mathfrak{a}^{n_0} & \xrightarrow{\delta_1} & \operatorname{Ext}^1_R(R/\mathfrak{a}^{n_0}, H^{t-1}_{\mathfrak{a}}(M)^2) \\
\downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle id} \\
(0 :_{H^t_{\mathfrak{a}}(\overline{M})} \mathfrak{a}^{n_0})^2 & \xrightarrow{\delta_2} & \operatorname{Ext}^1_R(R/\mathfrak{a}^{n_0}, H^{t-1}_{\mathfrak{a}}(M)^2),
\end{array}$$

where δ_1, δ_2 are connected homomorphisms. Because $F^{t-1}_{n_0,x}$, $F^{t-1}_{n_0,y}$ are determined, $\delta_2 = 0$, so $\delta_1 = 0$. Hence we obtain the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 :_{H^{t-1}_{\mathfrak{a}}(M)^2} \mathfrak{a}^{n_0} & \longrightarrow & 0 :_{H^{t-1}_{\mathfrak{a}}(N)} \mathfrak{a}^{n_0} & \longrightarrow & 0 :_{H^t_{\mathfrak{a}}(\overline{M})} \mathfrak{a}^{n_0} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle id} & & \downarrow & & \downarrow{\scriptstyle \Delta} & & \\
0 & \longrightarrow & 0 :_{H^{t-1}_{\mathfrak{a}}(M)^2} \mathfrak{a}^{n_0} & \longrightarrow & 0 :_{H^{t-1}_{\mathfrak{a}}(M/xM) \oplus H^{t-1}_{\mathfrak{a}}(M/yM)} \mathfrak{a}^{n_0} & \longrightarrow & 0 :_{H^t_{\mathfrak{a}}(\overline{M})^2} \mathfrak{a}^{n_0} & \longrightarrow & 0.
\end{array}$$

The sequence in the bottom row is just $F^{t-1}_{n_0,x} \oplus F^{t-1}_{n_0,y}$. We denote the sequence in the top row by $F^{t-1}_{n_0}$, so

$$F^{t-1}_{n_0} = \left(F^{t-1}_{n_0,x} \oplus F^{t-1}_{n_0,y}\right)\Delta_{0:_{H^t_{\mathfrak{t}}(\overline{M})}\mathfrak{a}^{n_0}}. \tag{2}$$

On the other hand, we consider the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \overline{M} & \xrightarrow{\overline{\varphi}} & M \oplus M & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \nabla_M} & & \downarrow & & \\
0 & \longrightarrow & \overline{M} & \xrightarrow{x+y} & M & \longrightarrow & M/(x+y)M & \longrightarrow & 0,
\end{array}$$

where $\nabla_M : M \oplus M \to M$, $\nabla(m, m') = m + m'$ is the codiagonal homomorphism. Since derived homomorphisms of ∇_M are also codiagonal homomorphisms, so is the homomorphism $\nabla_{H^i_{\mathfrak{a}}(M)} : H^i_{\mathfrak{a}}(M) \oplus H^i_{\mathfrak{a}}(M) \to H^i_{\mathfrak{a}}(M)$ for all $i \geqslant 0$. Hence by applying the functor $H^i_{\mathfrak{a}}(\bullet)$ to the above diagram we get the following commutative diagram

$$\begin{array}{ccccccccc}
E^i : 0 & \longrightarrow & H^i_{\mathfrak{a}}(M) \oplus H^i_{\mathfrak{a}}(M) & \longrightarrow & H^i_{\mathfrak{a}}(N) & \longrightarrow & H^{i+1}_{\mathfrak{a}}(\overline{M}) & \longrightarrow & 0 \\
 & & \Big\downarrow {\scriptstyle \nabla_{H^i_{\mathfrak{a}}(M)}} & & \Big\downarrow & & \Big\downarrow {\scriptstyle id} & & \\
E^i_{x+y} : 0 & \longrightarrow & H^i_{\mathfrak{a}}(M) & \longrightarrow & H^i_{\mathfrak{a}}(M/(x+y)M) & \longrightarrow & H^{i+1}_{\mathfrak{m}}(\overline{M}) & \longrightarrow & 0,
\end{array}$$

for all $i < t-1$. It follows for all $i < t-1$ that

$$E^i_{x+y} = \nabla_{H^i_{\mathfrak{m}}(M)} E^i. \tag{3}$$

Moreover, if $H^t_{\mathfrak{a}}(M) \cong H^t_{\mathfrak{a}}(\overline{M})$, we have

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^{t-1}_{\mathfrak{a}}(M)^2 & \longrightarrow & H^{t-1}_{\mathfrak{a}}(N) & \longrightarrow & K_{(x,y)} & \longrightarrow & 0 \\
 & & \Big\downarrow {\scriptstyle \nabla_{H^{t-1}_{\mathfrak{a}}(M)}} & & \Big\downarrow & & \Big\downarrow {\scriptstyle \mu} & & \\
0 & \longrightarrow & H^{t-1}_{\mathfrak{a}}(M) & \longrightarrow & H^{t-1}_{\mathfrak{a}}(M/(x+y)M) & \longrightarrow & K_{x+y} & \longrightarrow & 0,
\end{array}$$

where μ is injective. By applying the functor $\mathrm{Hom}_R(R/\mathfrak{a}^{n_0}, \bullet)$ to the above diagram we get

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (0 :_{H^{t-1}_{\mathfrak{a}}(M)} \mathfrak{a}^{n_0})^2 & \longrightarrow & 0 :_{H^{t-1}_{\mathfrak{a}}(N)} \mathfrak{a}^{n_0} & \longrightarrow & 0 :_{H^t_{\mathfrak{a}}(\overline{M})} \mathfrak{a}^{n_0} & \longrightarrow & 0 \\
 & & \Big\downarrow {\scriptstyle \nabla_{0 :_{H^{t-1}_{\mathfrak{a}}(M)} \mathfrak{a}^{n_0}}} & & \Big\downarrow & & \Big\downarrow {\scriptstyle id} & & \\
0 & \longrightarrow & 0 :_{H^{t-1}_{\mathfrak{a}}(M)} \mathfrak{a}^{n_0} & \longrightarrow & 0 :_{H^{t-1}_{\mathfrak{m}}(M/(x+y)M)} \mathfrak{a}^{n_0} & \longrightarrow & 0 :_{H^t_{\mathfrak{a}}(\overline{M})} \mathfrak{a}^{n_0}. & &
\end{array}$$

It follows from the existence of $F^{t-1}_{n_0}$ that the bottom row is exact, and hence $F^{t-1}_{n_0,x+y}$ is determined. Therefore

$$F^{t-1}_{n_0,x+y} = \nabla_{0 :_{H^{t-1}_{\mathfrak{a}}(M)} \mathfrak{a}^{n_0}} F^{t-1}_{n_0}. \tag{4}$$

Combining (1) and (3), we have

$$E^i_{x+y} = \nabla_{H^i_{\mathfrak{a}}(M)} \big(E^i_x \oplus E^i_y\big) \Delta_{H^{i+1}_{\mathfrak{a}}(\overline{M})},$$

for all $i < t-1$. So $E^i_{x+y} = E^i_x + E^i_y$ for all $i < t-1$.

Combining (2) and (4), we have

$$F^{t-1}_{n_0,x+y} = \nabla_{0 :_{H^{t-1}_{\mathfrak{a}}(M)} \mathfrak{a}^{n_0}} \big(F^{t-1}_{n_0,x} \oplus F^{t-1}_{n_0,y}\big) \Delta_{0 :_{H^t_{\mathfrak{a}}(\overline{M})} \mathfrak{a}^{n_0}}.$$

So $F^{t-1}_{n_0,x+y} = F^{t-1}_{n_0,x} + F^{t-1}_{n_0,y}$.

(ii) The first and second claims of (ii) can be shown by the same method as used in the proof of Proposition 2.1, and the last one follows immediately from the structure of R-module of the extension group Ext^1. $\square$

3. The proof of the main result

First of all, we need some auxiliary lemmas.

Lemma 3.1. *Let $(R,\mathfrak{m})$ be a Noetherian local ring, $\mathfrak{a}$, $\mathfrak{b}$ ideals and $\mathfrak{p}_1,\dots,\mathfrak{p}_n$ prime ideals such that $\mathfrak{a}\mathfrak{b}\nsubseteq\mathfrak{p}_j$ for all $j\leqslant n$. Let x be an element contained in $\mathfrak{a}\mathfrak{b}$ but $x\notin\mathfrak{p}_j$ for all $j\leqslant n$. Then there are elements $a_1,\dots,a_r\in\mathfrak{a}$, $b_1,\dots,b_r\in\mathfrak{b}$ that we can write $x=a_1b_1+\dots+a_rb_r$ such that $a_ib_i\notin\mathfrak{p}_j$ and $a_1b_1+\dots+a_ib_i\notin\mathfrak{p}_j$ for all $i\leqslant r$, $j\leqslant n$.*

Proof. It is sufficient to prove the assertion in the case $\mathfrak{p}_i\nsubseteq\mathfrak{p}_j$ for all $i,j\leqslant n, i\neq j$. By the Prime Avoidance Theorem, we can choose a system of generators $a_1b_1,\dots,a_rb_r$ of $\mathfrak{a}\mathfrak{b}$ such that $a_i\in\mathfrak{a}$, $b_i\in\mathfrak{b}$ for all $i\leqslant r$, and $a_ib_i\notin\mathfrak{p}_j$ for all $i\leqslant r, j\leqslant n$. Hence there exist $s_i\in R$, $i=1,\dots,r$, such that $x=s_1a_1b_1+\dots+s_ra_rb_r$. Rewrite $x=a_1(s_1b_1)+\dots+a_r(s_rb_r)$, therefore we may assume without loss of generality that x can be written in form $x=a_1b_1+a_2b_2+\dots+a_rb_r$ with $a_i\in\mathfrak{a}$, $b_i\in\mathfrak{b}$ for all $i\leqslant r$, and $a_i\notin\mathfrak{p}_j$ for all $i\leqslant r, j\leqslant n$.

We prove the assertion by induction on r. The case $r=1$ is trivial. Assume that $r>1$ and the lemma is true for $r-1$. Set $J=\{j\colon b_r\in\mathfrak{p}_j\}$. By the Prime Avoidance Theorem we can choose $u\in\mathfrak{b}$ such that $u\notin\mathfrak{p}_j$ for all $j\in J$, and $u\in\mathfrak{p}_j$ for all $j\notin J$. Since $a_1\notin\mathfrak{p}_j$ for all $j\leqslant n$, ua_1 also has this property. Therefore $b_r+ua_1\notin\mathfrak{p}_j$ for all $j\leqslant n$. We write $x=a_1(b_1-ua_r)+a_2b_2+\dots+a_r(b_r+ua_1)$, so without loss of generality we can assume more that $x=a_1b_1+a_2b_2+\dots+a_rb_r$ and $a_rb_r\notin\mathfrak{p}_j$ for all $j\leqslant n$. Let $x'=a_1b_1+\dots+a_{r-1}b_{r-1}$, and set $J'=\{j\colon x'\in\mathfrak{p}_j\}$. Using the Prime Avoidance Theorem again we can choose $v\in\mathfrak{m}$ such that $v\notin\mathfrak{p}_j$ for all $j\in J'$, and $v\in\mathfrak{p}_j$ for all $j\notin J'$. Because $a_1,a_r,b_r\notin\mathfrak{p}_j$ for all $j\leqslant n$, $va_1a_rb_r$ has the same property as v. Set $x_{r-1}=x'+va_1a_rb_r=a_1(b_1+va_rb_r)+a_2b_2+\dots+a_{r-1}b_{r-1}$ Then $x_{r-1}\notin\mathfrak{p}_j$ for all $j\leqslant n$ and $x=x_{r-1}+a_rb_r(1-va_1)$. Since $a_rb_r(1-va_1)\notin\mathfrak{p}_j$ for all $j\leqslant n$, the conclusion follows from the inductive hypothesis for the element x_{r-1}. $\square$

Corollary 3.2. *Let $(R,\mathfrak{m})$ be a Noetherian local ring and $\mathfrak{a}$ an ideal of R. Let M be a finitely generated R-module and $x\in\mathfrak{a}^2$ an $\mathfrak{a}$-filter regular element of M. Then we can find $\mathfrak{a}$-filter regular elements $a_1,\dots,a_r$, $b_1,\dots,b_r\in\mathfrak{a}$ of M such that $x=a_1b_1+\dots+a_rb_r$ and $a_1b_1+\dots+a_ib_i$ are also $\mathfrak{a}$-filter regular elements of M for all $i\leqslant r$.*

Proof. It follows from Lemma 3.1 with $\mathfrak{a}=\mathfrak{b}$ and $\mathrm{Ass}(M)\setminus V(\mathfrak{a})$ for the set of prime ideals $\{\mathfrak{p}_1,\dots,\mathfrak{p}_n\}$. $\square$

The following result is somehow known. The proof of this lemma follows easily from the commutativity of localizations and the functor Hom of finitely generated modules, therefore we omit it.

Lemma 3.3. *Let A, B, C are finitely generated R-modules. Then the sequence*

$$0\longrightarrow A\longrightarrow B\longrightarrow C\longrightarrow 0$$

is a split exact sequence if and only if the sequence

$$0\longrightarrow A_{\mathfrak{m}}\longrightarrow B_{\mathfrak{m}}\longrightarrow C_{\mathfrak{m}}\longrightarrow 0$$

is exact and split for all maximal ideal $\mathfrak{m}$ of R.

Proof of Theorem 1.1. Keep all notations as in Section 3 with $U = H^0_{\mathfrak{a}}(M)$. Then every $\mathfrak{a}$-filter regular element $x \in \mathfrak{a}^{n_0}$ satisfies the condition (♯). Since $H^i_{\mathfrak{a}}(M) \cong H^i_{\mathfrak{a}}(\overline{M})$ for all $i > 0$, $H^i_{\mathfrak{a}}(\overline{M})$ is finitely generated, so is $0 :_{H^t_{\mathfrak{a}}(\overline{M})} \mathfrak{a}^{n_0}$ by Theorem 1.2, [1]. Using localizations at maximal ideals we may assume by Lemma 3.3 that $(R, \mathfrak{m})$ is a Noetherian local ring. Let $x \in \mathfrak{a}^{2n_0}$. There are by Corollary 3.2 $\mathfrak{a}$-filter regular elements $a_i, b_i \in \mathfrak{a}^{n_0}$, $i \leqslant r$ such that $x = a_1b_1 + \cdots + a_rb_r$ and $a_1b_1 + \cdots + a_jb_j$ are $\mathfrak{a}$-filter regular elements for all $1 \leqslant j \leqslant r$. Then, by virtue of Theorem 2.2(i) we have

$$E^i_x = E^i_{a_1b_1+\cdots+a_rb_r} = E^i_{a_1b_1} + E^i_{a_2b_2} + \cdots + E^i_{a_rb_r}.$$

Therefore

$$E^i_x = a_1E^i_{b_1} + a_2E^i_{b_2} + \cdots + a_rE^i_{b_r} = 0$$

by Theorem 2.2(ii) for all $0 \leqslant i < t - 1$. Thus we have

$$H^i_{\mathfrak{a}}(M/xM) \cong H^i_{\mathfrak{a}}(M) \oplus H^{i+1}_{\mathfrak{a}}(\overline{M}) \cong H^i_{\mathfrak{a}}(M) \oplus H^{i+1}_{\mathfrak{a}}(M)$$

for all $0 \leqslant i < t - 1$. On the other hand, by Proposition 2.1 $F^{t-1}_{n_0,a_jb_j}$ are determined for all $j \leqslant r$. It follows by Theorem 2.2(i) that $F^{t-1}_{n_0,x} = F^{t-1}_{n_0,a_1b_1+\cdots+a_rb_r}$ is determined and

$$F^{t-1}_{n_0,x} = F^{t-1}_{n_0,a_1b_1} + \cdots + F^{t-1}_{n_0,a_rb_r}.$$

Therefore $F^{t-1}_{n_0,x} = 0$ by Theorem 2.2(ii), so

$$0 :_{H^{t-1}_{\mathfrak{a}}(M/xM)} \mathfrak{a}^{n_0} \cong H^{t-1}_{\mathfrak{a}}(M) \oplus 0 :_{H^t_{\mathfrak{a}}(M)} \mathfrak{a}^{n_0},$$

since $0 :_{H^{t-1}_{\mathfrak{a}}(M)} \mathfrak{a}^{n_0} = H^{t-1}_{\mathfrak{a}}(M)$. □

4. Some applications

The first immediate consequence of Theorem 1.1 is an affirmative complete answer for the question posed in the introduction.

Corollary 4.1. *Let M be a generalized Cohen–Macaulay module over a local ring $(R, \mathfrak{m})$ of dimension $d > 0$, and n_0 the least positive integer such that $\mathfrak{m}^{n_0} H^i_{\mathfrak{m}}(M) = 0$ for all $i < d$. Then for any parameter element $x \in \mathfrak{m}^{2n_0}$, we have*

$$H^i_{\mathfrak{m}}(M/xM) \cong H^i_{\mathfrak{m}}(M) \oplus H^{i+1}_{\mathfrak{m}}(M),$$

for all $i < d - 1$, and

$$0 :_{H^{d-1}_{\mathfrak{m}}(M/xM)} \mathfrak{m}^{n_0} \cong H^{d-1}_{\mathfrak{m}}(M) \oplus 0 :_{H^d_{\mathfrak{m}}(M)} \mathfrak{m}^{n_0}.$$

The next application of Theorem 1.1 is somehow strange to the authors and it can be used to derive many consequences.

Corollary 4.2. *Let M be a finitely generated module over a Noetherian ring R and $\mathfrak{a}$ an ideal of R. Let t and n_0 be positive integers such that $\mathfrak{a}^{n_0} H^i_{\mathfrak{a}}(M) = 0$ for all $i < t$. Let $x_1, \dots, x_t$ be an $\mathfrak{a}$-filter regular sequence of M contained in $\mathfrak{a}^{2n_0}$. Then for all positive integer $k \leqslant n_0$ and all $j = 1, \dots, t$, $\operatorname{Hom}_R(R/\mathfrak{a}^k, M/(x_1, \dots, x_j)M)$ are independent of the choice of the sequence $x_1, \dots, x_j$. Moreover, we have*

$$\operatorname{Hom}_R\big(R/\mathfrak{a}^k, M/(x_1, \dots, x_j)M\big) \cong \bigoplus_{i=0}^{j} \operatorname{Hom}_R\big(R/\mathfrak{a}^k, H^i_{\mathfrak{a}}(M)\big)^{\binom{j}{i}}.$$

Proof. We proceed by induction on j. From Theorem 1.1 we have

$$\operatorname{Hom}_R\big(R/\mathfrak{a}^k, H^i_{\mathfrak{a}}(M/x_1M)\big) \cong \operatorname{Hom}_R\big(R/\mathfrak{a}^k, H^i_{\mathfrak{a}}(M)\big) \oplus \operatorname{Hom}_R\big(R/\mathfrak{a}^k, H^{i+1}_{\mathfrak{a}}(M)\big)$$

for all $i \leqslant t-1$. Therefore

$$\begin{aligned}\operatorname{Hom}_R\big(R/\mathfrak{a}^k, M/(x_1)M\big) &\cong \operatorname{Hom}_R\big(R/\mathfrak{a}^k, H^0_{\mathfrak{a}}(M/x_1M)\big)\\ &\cong \operatorname{Hom}_R\big(R/\mathfrak{a}^k, H^0_{\mathfrak{a}}(M)\big) \oplus \operatorname{Hom}_R\big(R/\mathfrak{a}^k, H^1_{\mathfrak{a}}(M)\big),\end{aligned}$$

and the corollary is proved for $j = 1$. Suppose that $j > 1$. By Theorem 1.1 we have $\mathfrak{a}^{n_0} H^i_{\mathfrak{a}}(M/x_1M) = 0$ for all $i < t-1$. It follows from the inductive hypothesis for the sequence $x_2, \dots, x_j$ and M/x_1M that

$$\begin{aligned}\operatorname{Hom}_R\big(R/\mathfrak{a}^k, M/(x_1, \dots, x_j)M\big) &\cong \bigoplus_{i=0}^{j-1} \operatorname{Hom}_R\big(R/\mathfrak{a}^k, H^i_{\mathfrak{a}}(M/x_1M)\big)^{\binom{j-1}{i}}\\ &\cong \bigoplus_{i=0}^{j} \operatorname{Hom}_R\big(R/\mathfrak{a}^k, H^i_{\mathfrak{a}}(M)\big)^{\binom{j}{i}}\end{aligned}$$

as required. □

Let M be a finitely generated module over a Noetherian local ring $(R, \mathfrak{m})$ and $\mathfrak{q}$ a parameter ideal of M. The index of irreducibility of $\mathfrak{q}$ on M is defined by $N_R(\mathfrak{q}, M) = \dim_{R/\mathfrak{m}} \operatorname{Soc}(M/\mathfrak{q}M)$, where $\operatorname{Soc}(N) \cong 0 :_N \mathfrak{m} \cong \operatorname{Hom}(R/\mathfrak{m}, N)$ for an arbitrary R-module N. It is well known that if M is a Cohen–Macaulay module then $N_R(\mathfrak{q}, M)$ is a constant independent of the choice of $\mathfrak{q}$. In the case M is a Buchsbaum module, S. Goto and H. Sakurai proved in [5] that for large enough n the index of irreducibility $N_R(\mathfrak{q}, M)$ is a constant for all parameter ideals $\mathfrak{q}$ contained in $\mathfrak{m}^n$. And they conjectured that this result is also true for generalized Cohen–Macaulay modules. H.L. Truong and the first author have given an affirmative answer for this conjecture in [4]. Now, in virtue of Corollary 4.2 we can prove a statement which is a slight generalization of the main result of [4] as follows.

Corollary 4.3. *Let M be a generalized Cohen–Macaulay module of dimension d and n_0 a positive integer such that $\mathfrak{m}^{n_0} H^i_{\mathfrak{m}}(M) = 0$ for all $i < d$. Then, for every parameter ideal $\mathfrak{q}$ of M contained in $\mathfrak{m}^{2n_0}$ and $k \leqslant n_0$, the length $\ell_R((\mathfrak{q}M :_M \mathfrak{m}^k)/\mathfrak{q}M)$ is independent of the choice of $\mathfrak{q}$ and given by*

$$\ell_R\big((\mathfrak{q}M :_M \mathfrak{m}^k)/\mathfrak{q}M\big) = \sum_{i=0}^{d} \binom{d}{i} \ell_R\big(0 :_{H^i_{\mathfrak{m}}(M)} \mathfrak{m}^k\big).$$

In particular, the index of irreducibility $N_R(\mathfrak{q}, M)$ is a constant independent of the choice of $\mathfrak{q}$ and

$$N_R(\mathfrak{q}, M) = \sum_{i=0}^{d} \binom{d}{i} \dim_{R/\mathfrak{m}} \operatorname{Soc}\bigl(H^i_{\mathfrak{m}}(M)\bigr).$$

Proof. It follows immediately from Corollary 4.2 and the fact that $\operatorname{Hom}_R(R/\mathfrak{m}^k, M/\mathfrak{q}M) \cong (\mathfrak{q}M :_M \mathfrak{m}^k)/\mathfrak{q}M$ and $\operatorname{Hom}_R(R/\mathfrak{m}^k, H^i_{\mathfrak{m}}(M)) \cong 0 :_{H^i_{\mathfrak{m}}(M)} \mathfrak{m}^k$ for all i. $\square$

In [6] C. Huneke conjectured that the set $\operatorname{Ass} H^i_{\mathfrak{a}}(M)$ is a finite set for any ideal $\mathfrak{a}$ and all i. The conjecture was settled by G. Lyubeznik [9] and C. Huneke, R.Y. Sharp [7] for regular local rings containing a field. Although M. Katzman [8] has given an example of a Noetherian ring and an ideal $\mathfrak{a}$ such that $H^2_{\mathfrak{a}}(R)$ has infinitely many associated primes, the conjecture is still true in many interesting cases. The following result is an immediate consequence of Corollary 4.2, which is an extension of the main result of M. Brodmann and A.L. Faghani in [2].

Corollary 4.4. *Let M be a finitely generated R-module and $\mathfrak{a}$ an ideal of R. Let t and n_0 be positive integers such that $\mathfrak{a}^{n_0} H^i_{\mathfrak{a}}(M) = 0$ for all $i < t$. Then for every $\mathfrak{a}$-filter regular sequence $x_1, \ldots, x_t$ of M contained in $\mathfrak{a}^{2n_0}$, we have*

$$\bigcup_{i=0}^{j} \operatorname{Ass} H^i_{\mathfrak{a}}(M) = \operatorname{Ass}\bigl(M/(x_1, \ldots, x_j M)\bigr) \bigcap V(\mathfrak{a})$$

for all $j = 1, \ldots, t$. In particular, $H^t_{\mathfrak{a}}(M)$ has only finitely many associated primes.

Proof. Since $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-torsion, $\operatorname{Ass} H^i_{\mathfrak{a}}(M) = \operatorname{Ass} \operatorname{Hom}_R(R/\mathfrak{a}, H^i_{\mathfrak{a}}(M))$. It follows from Corollary 4.2 that for all $j = 1, \ldots, t$,

$$\bigcup_{i=0}^{j} \operatorname{Ass} H^i_{\mathfrak{a}}(M) = \operatorname{Ass} H^0_{\mathfrak{a}}\bigl(M/(x_1, \ldots, x_j)M\bigr) = \operatorname{Ass}\bigl(M/(x_1, \ldots, x_j)M\bigr) \cap V(\mathfrak{a}). \qquad \square$$

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