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ON THE FINITENESS OF ASSOCIATED PRIMES OF LOCAL COHOMOLOGY MODULES

PHAM HUNG QUY

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ABSTRACT. Let R be a Noetherian ring, $\mathfrak{a}$ be an ideal of R and M be a finitely generated R-module. The aim of this paper is to show that if t is the least integer such that neither $H^t_{\mathfrak{a}}(M)$ nor $\operatorname{supp}(H^t_{\mathfrak{a}}(M))$ is non-finite, then $H^t_{\mathfrak{a}}(M)$ has finitely many associated primes. This combines the main results of Brodmann and Faghani and independently of Khashyarmanesh and Salarian.

1. Introduction

Throughout this paper, R is a Noetherian ring (with identity), $\mathfrak{a}$ is an ideal of R and M is an R-module. For basic facts about commutative algebra see [3] and [8]; for local cohomology refer to [2]. A module is finite if it is finitely generated and a set is finite if it has finitely many elements. We use $\mathbb{N}_0$ to denote the set of non-negative integers.

An interesting problem in commutative algebra is determining when the set of associated primes of the ith local cohomology module $H^i_{\mathfrak{a}}(M)$ of M is finite. If R is a regular local ring containing a field, then $H^t_{\mathfrak{a}}(R)$ has only finitely many associated primes for all $i \geq 0$; cf. [4] (in positive characteristic) and [7] (in characteristic zero). However, Katzman [5] has given an example of a Noetherian local ring and an ideal $\mathfrak{a}$ such that $H^2_{\mathfrak{a}}(R)$ has infinitely many associated primes. But we have many interesting results about the finiteness of $\operatorname{Ass}_R(H^t_{\mathfrak{a}}(M))$. It is well known that if M is finite, then $\operatorname{Ass}_R(H^t_{\mathfrak{a}}(M))$ is finite in either of the following cases:

(a) $H^i_{\mathfrak{a}}(M)$ is finite for all $i < t$; see [1] and [6];

(b) $\operatorname{supp}(H^t_{\mathfrak{a}}(M))$ is finite for all $i < t$; see [6].

The aim of this paper is to combine (a) and (b). That is, *if M is finitely generated, then $H^t_{\mathfrak{a}}(M)$ has only finitely many associated primes if, for all $i < t$, $H^i_{\mathfrak{a}}(M)$ is finite or has finite support.*

In section 2, we define: M is an *FSF* module if there is a finite submodule N of M such that the quotient module M/N has finite support, and we give some properties of FSF modules.

In section 3, we will prove the following: *Let $\mathfrak{a}$ be an ideal of the Noetherian ring R, and let M be an FSF R-module. Let $t \in \mathbb{N}_0$ be such that $H^i_{\mathfrak{a}}(M)$ is FSF for all $i < t$. Then $\operatorname{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M))$ is FSF. Therefore, $\operatorname{Ass}_R(H^t_{\mathfrak{a}}(M))$ is finite.* This implies the main result as a consequence.

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2. FSF module

Definition 2.1. Let R be a Noetherian ring and M be an R-module. M is called an *FSF* module if there is a **F**inite submodule N of M such that **S**upport of the quotient module M/N is **F**inite.

Proposition 2.2. *Let M be an R-module. We have*

(i) *If M is an FSF module, then $\mathrm{Ass}_R(M)$ is finite.*

(ii) *Let $0 \to M_1 \to M \to M_2 \to 0$ be an exact sequence of R-modules. Then M is FSF iff M_1 and M_2 are FSF.*

(iii) *Let M be an FSF module and N be finite. Then $\mathrm{Ext}^i_R(N, M)$ is FSF for all $i \geq 0$.*

Proof. (i). This is trivial from the definition of FSF modules.

(ii). "$\Rightarrow$." If M is an FSF module, it is easy to show that M_1 and M_2 are FSF.

"$\Leftarrow$." Suppose that M_1 and M_2 are FSF. Let N_1 and N_2 be finitely generated submodules of M_1 and M_2, respectively, such that $\mathrm{supp}(M_1/N_1)$ and $\mathrm{supp}(M_2/N_2)$ are finite. We may assume that M_1 is a submodule of M and that M_2 is a quotient module of M. Let $x_1, x_2, ..., x_n, y_1, y_2, ..., y_m$ in M such that $x_1, x_2, ..., x_n$ are generators of N_1 and $\overline{y}_1, \overline{y}_2, ..., \overline{y}_m$ are generators of N_2 in $M_2 = M/M_1$. Let N be a submodule of M generated by $x_1, x_2, ..., x_n, y_1, y_2, ..., y_m$, so N is finite, and it is not difficult to show that $\mathrm{supp}(M/N)$ is finite. Hence, M is FSF.

(iii) M is FSF, so there exists an exact sequence

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0,$$

with M_1 finitely generated and $\mathrm{supp}(M_2)$ finite. This exact sequence induces exact sequences

$$\mathrm{Ext}^i_R(N, M_1) \longrightarrow \mathrm{Ext}^i_R(N, M) \longrightarrow \mathrm{Ext}^i_R(N, M_2)$$

for all $i \in \mathbb{N}_0$. Since N and M_1 are finitely generated modules and $\mathrm{supp}(M_2)$ is finite, we have that $\mathrm{Ext}^i_R(N, M_1)$ is finitely generated and $\mathrm{supp}(\mathrm{Ext}^i_R(N, M_2))$ is finite. Hence, $\mathrm{Ext}^i_R(N, M)$ is FSF for all $i \in \mathbb{N}_0$. $\square$

3. The main result

Proposition 3.1. *Let $\mathfrak{a}$ be an ideal of the Noetherian ring R, and let M be an FSF R-module. Let $t \in \mathbb{N}_0$ be such that $H^i_{\mathfrak{a}}(M)$ is FSF for all $i < t$. Then*

$$\mathrm{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M))$$

is FSF. Therefore, $\mathrm{Ass}_R(H^t_{\mathfrak{a}}(M))$ is finite.

Proof. The last assertion follows from the first, from Proposition 2.2(i) and from the fact that $\mathrm{Ass}_R(H^t_{\mathfrak{a}}(M)) = \mathrm{Ass}_R(\mathrm{Hom}(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M)))$.

We prove that $\mathrm{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M))$ is FSF by induction on t. The case $t = 0$ is clear because $\mathrm{Hom}_R(R/\mathfrak{a}, H^0_{\mathfrak{a}}(M)) \subseteq M$.

So, let $t > 0$ and set $\overline{M} = M/H^0_{\mathfrak{a}}(M)$. Then $\overline{M}$ is FSF, $H^0_{\mathfrak{a}}(\overline{M}) = 0$, and

$$H^k_{\mathfrak{a}}(\overline{M}) \cong H^k_{\mathfrak{a}}(M)$$

for all $k > 0$. Thus $H^i_{\mathfrak{a}}(\overline{M})$ is FSF for all $i < t$ and $H^t_{\mathfrak{a}}(\overline{M}) \cong H^t_{\mathfrak{a}}(M)$. Replace M by $\overline{M}$ and assume henceforth that $H^0_{\mathfrak{a}}(M) = 0$. By Proposition 2.2(i), we have that $\mathrm{Ass}_R(M)$ is finite. Combining this with $H^0_{\mathfrak{a}}(M) = 0$ implies that there exists $a \in \mathfrak{a}$ such that a is an M-regular element. So, we have the short exact sequence

$$0 \longrightarrow M \xrightarrow{a \cdot} M \xrightarrow{p} M/aM \longrightarrow 0,$$

where p is natural projection. This yields the exact cohomology sequences

$$H^i_{\mathfrak{a}}(M) \longrightarrow H^i_{\mathfrak{a}}(M/aM) \longrightarrow H^{i+1}_{\mathfrak{a}}(M) \quad (\forall i \in \mathbb{N}_0).$$

Hence, $H^i_{\mathfrak{a}}(M/aM)$ is FSF for all $i < t-1$. It is clear that M/aM is FSF, so by induction, we have that $\operatorname{Hom}_R(R/\mathfrak{a}, H^{t-1}_{\mathfrak{a}}(M/aM))$ is FSF.

We consider the long exact sequence

$$(*) \qquad H^{t-1}_{\mathfrak{a}}(M) \xrightarrow{a\cdot} H^{t-1}_{\mathfrak{a}}(M) \xrightarrow{H^{t-1}_{\mathfrak{a}}(p)} H^{t-1}_{\mathfrak{a}}(M/aM) \longrightarrow H^t_{\mathfrak{a}}(M) \xrightarrow{a\cdot} H^t_{\mathfrak{a}}(M).$$

Let $N = \frac{H^{t-1}_{\mathfrak{a}}(M)}{aH^{t-1}_{\mathfrak{a}}(M)}$ and $N' = \operatorname{coker}(H^{t-1}_{\mathfrak{a}}(p))$. We split the exact sequence $(*)$ into two exact sequences:

$$(**) \qquad 0 \longrightarrow N \longrightarrow H^{t-1}_{\mathfrak{a}}(M/aM) \longrightarrow N' \longrightarrow 0,$$

$$(***) \qquad 0 \longrightarrow N' \longrightarrow H^t_{\mathfrak{a}}(M) \xrightarrow{a\cdot} H^t_{\mathfrak{a}}(M).$$

From sequence $(**)$ we deduce that the sequence

$$\operatorname{Hom}_R(R/\mathfrak{a}, H^{t-1}_{\mathfrak{a}}(M/aM)) \longrightarrow \operatorname{Hom}_R(R/\mathfrak{a}, N') \longrightarrow \operatorname{Ext}^1_R(R/\mathfrak{a}, N)$$

is exact. The left-most module is FSF as above and the right-most module is FSF by Proposition 2.2(iii); therefore, $\operatorname{Hom}_R(R/\mathfrak{a}, N')$ is FSF. Furthermore, $(***)$ gives the exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(R/\mathfrak{a}, N') \longrightarrow \operatorname{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M)) \xrightarrow{a\cdot} \operatorname{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M)).$$

On the other hand, the multiplication homomorphism

$$a\cdot :\ \operatorname{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M)) \to \operatorname{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M))$$

is zero since $a \in \mathfrak{a}$.

So, we have that $\operatorname{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M)) \cong \operatorname{Hom}_R(R/\mathfrak{a}, N')$ is FSF, as desired. □

Finally, we have

Theorem 3.2. *Let $\mathfrak{a}$ be an ideal of the Noetherian ring R, and let M be a finitely generated R-module. Let $t \in \mathbb{N}_0$ be such that either $H^i_{\mathfrak{a}}(M)$ is finite or $\operatorname{supp}(H^i_{\mathfrak{a}}(M))$ is finite for all $i < t$. Then $\operatorname{Ass}_R(H^t_{\mathfrak{a}}(M))$ is finite.*

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Department of Mathematics, FPT University (Dai Hoc FPT), 15B Pham Hung Street, Ha Noi, Vietnam
E-mail address: phamhungquy@gmail.com
E-mail address: quyph@fpt.edu.vn