On Delayed Logistic Equation Driven by Fractional Brownian Motion

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ABSTRACT

In this paper we use the fractional stochastic integral given by Carmona et al. [1] to study a delayed logistic equation driven by fractional Brownian motion which is a generalization of the classical delayed logistic equation . We introduce an approximate method to find the explicit expression for the solution. Our proposed method can also be applied to the other models and to illustrate this, two models in physiology are discussed.

1 Introduction

The classical logistic equation was proposed by Verhulst [2] to describe population growth in a limited environment and until now it has still been very popular in population dynamics. However, in some cases where there is a lag in some of the processes involved Hutchinson [3] pointed out that the logistic equation would be inappropriate for the description of population growth. As an example, he introduced the delayed logistic equation (or Wright' equation)

$$dX_t = (a - bX_{t-\tau})X_t dt, \tag{1}$$

to model a single population whose per capita rate of growth at time t.

When modeling systems which do not noticeably affect their environment, stochastic processes are often used to model the environmental fluctuations, thus leading to stochastic differential equations with or without delay. In the case where noise is a Brownian motion, Alvarez and Shepp [4] and Guillouzic [5] studied the following stochastic logistic equations, respectively

$$dX_t = (a - bX_t)X_t dt + \sigma X_t dW_t \ , \ X_0 = x > 0, \tag{2}$$

$$dX_t = (a - bX_{t-\tau})X_t dt + \sigma X_t dW_t \ , \tag{3}$$

where W_t is a standard Brownian motion.

In the last decades there has been an increased interest in stochastic models based on other processes rather than the Brownian motion, much of the literature has pointed out that fractional Brownian motion (or fractal noise) provides a natural theoretical framework to model many phenomena arising in biology (for example, see [6, 7]). This naturally leads us to investigate logistic equations driven by fractional Brownian motion, that is to replace Brownian motion in equations (2), (3) by a fractional Brownian motion. In [8] the author solved the stochastic differential equation with polynomial drift and fractal noise which is a generalization of logistic equation driven by fractional Brownian motion

$$dX_t = (aX_t - bX_t^n)dt + \sigma X_t dW_t^H \tag{4}$$

The study of the stochastic differential equations depends on the definitions of the stochastic integrals involved. Among the many definitions of the fractional stochastic integral we choose, in this paper, a definition given by Carmona et al. [1] to consider the delayed logistic equation driven by fractional Brownian motion

$$\begin{cases} dX_t = (a - bX_{t-\tau})X_t dt + \sigma X_t dW_t^H \ , \ t \in [0,T], \\ X_t = \phi(t) \ , \ t \in [-r,0], \text{ where } \phi \in C[-r,0], \end{cases} \tag{5}$$

where W_t^H is a fractional Brownian motion (fBm) of the Liouville form with Hurst index $H \in (0,1)$ (see the definition in the following text), a, b are positive real numbers and σ is a real number. In the context of population dynamics, τ characterizes the reaction time of the population to environmental constraints, while b scales these constraints and a is the Malthusian growth rate.

Recently, Ferrante el at. studied some special forms of the delayed stochastic differential equations driven by fractional Brownian motion in [9, 10], however, all of them do not cover (5). Moreover, their method cannot be used in this paper because our equation (5) contains X_t in fractional stochastic integral.

This paper is organized as follows: In Section 2, we recall a definition of fractional stochastic integral given in [1]. Section 3 contains the main results of this paper that we propose an approximate method to find the explicit solution to the equation (5). In Section 4, we discuss some other models in physiology to more clearly see advantages of the approximate method used in previous section. The conclusion is given in Section 5.

2 Preliminaries

In this section we recall some fundamental results about the method of approximate fBm by semimartingales and a definition of stochastic integral with respect to fractional Brownian motion.

A fractional Brownian motion (fBm) of the Liouville form with Hurst index $H \in (0,1)$ is a centered Gaussian process defined by

$$W_t^H = \int_0^t K(t,s) dW_s \tag{6}$$

where W_s is a standard Brownian motion and the kernel $K(t,s) = (t-s)^\alpha$, $\alpha = H - \frac{1}{2}$.

It is well known that we cannot use classical Itô calculus to analyze models driven by fractional Brownian motion because fBm is neither semimartingale nor Markov process, except for the case where Hurst index $H = \frac{1}{2}$.

For every $\varepsilon > 0$ we define

$$W_t^{H,\varepsilon} = \int_0^t K(t+\varepsilon, s) dW_s = \int_0^t (t-s+\varepsilon)^\alpha dW_s. \tag{7}$$

From [11, 12] we know that $W_t^{H,\varepsilon}$ is a semimartingale with the following decomposition

$$W_t^{H,\varepsilon} = \varepsilon^\alpha W_t + \int_0^t \varphi_s^\varepsilon ds, \tag{8}$$

where $\varphi_s^\varepsilon = \int_0^s \alpha(s+\varepsilon-u)^{\alpha-1} dW_u$. Moreover, $W_t^{H,\varepsilon}$ converges in $L^p(\Omega)$, $p > 1$ uniformly in $t \in [0,T]$ to W_t^H as $\varepsilon \to 0$:

$$E|W_t^{H,\varepsilon} - W_t^H|^p \le c_{p,T} \varepsilon^{pH}.$$

Let us recall some elements of stochastic calculus of variations. For $h \in L^2([0,T], \mathbf{R})$, we denote by $W(h)$ the Wiener integral

$$W(h) = \int_0^T h(t) dW_t.$$

Let $\mathcal{S}$ denote the dense subset of $L^2(\Omega,\mathcal{F},P)$ consisting of those classes of random variables of the form

$$F = f(W(h_1),...,W(h_n)), \tag{9}$$

where $n\in\mathbf{N}, f\in C_b^\infty(\mathbf{R}^n, L^2([0,T],\mathbf{R})), h_1,...,h_n \in L^2([0,T],\mathbf{R})$. If F has the form (9), we define its derivative as the process $D^W F := \{D_t^W F, t\in[0,T]\}$ given by

$$D_t^W F = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(W(h_1),...,W(h_n))h_k(t).$$

For any $p\geq 1$ we shall denote by $\mathbf{D}_W^{1,p}$ the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,p} := \left[E|F|^p\right]^{\frac{1}{p}} + E\left[\int_0^T |D_u^W F|^p du\right]^{\frac{1}{p}}.$$

It is well known from [1, 13] that for an adapted process f belonging to the space $\mathbf{D}_W^{1,2}$ we have

$$\begin{aligned}\int_0^t f_s\,dW_s^{H,\varepsilon} &= \int_0^t f_s K(s+\varepsilon,s)\,dW_s + \int_0^t f_s\varphi_s^\varepsilon ds \\ &= \int_0^t f_s K(t+\varepsilon,s)\,dW_s + \int_0^t\int_s^t (f_u-f_s)\,\partial_1 K(u+\varepsilon,s)du\delta W_s + \int_0^t\int_0^u D_s^W f_u\,\partial_1 K(u+\varepsilon,s)dsdu\,,\end{aligned} \tag{10}$$

where the second integral in the right-hand side is a Skorokhod integral (we refer to [14] for more detail about the Skorohod integral).

Hypothesis (H): Assume that f is an adapted process belonging to the space $\mathbf{D}_W^{1,2}$ and that there exists β fulfilling $\beta+H>1/2$ and $p>1/H$ such that

(i) $\|f\|^2_{L_\beta^{1,2}} := \sup\limits_{0<s<u<T} \dfrac{E\left[(f_u-f_s)^2+\int_0^T (D_r^W f_u - D_r^W f_s)^2 dr\right]}{|u-s|^{2\beta}}$ is finite,

(ii) $\sup\limits_{0<s<T} |f_s|$ belongs to $L^p(\Omega)$.

In [1, 13] the authors proved that for $f\in$ **(H)**, $\int_0^t f_s\,dW_s^{H,\varepsilon}$ converges in $L^2(\Omega)$ as $\varepsilon\to 0$. Each term in the right-hand side of (10) converges to the same term where $\varepsilon = 0$. Then, it is "natural" to define

Definition 2.1. *Let $f\in$ **(H)**. The fractional stochastic integral of f with respect to W^H is defined by*

$$\int_0^t f_s\,dW_s^H = \int_0^t f_s K(t,s)\,dW_s + \int_0^t\int_s^t (f_u-f_s)\,\partial_1 K(u,s)du\delta W_s + \int_0^t du\int_0^u D_s^W f_u\,\partial_1 K(u,s)ds\,, \tag{11}$$

where $K(t,s) = (t-s)^\alpha, \partial_1 K(t,s) = \alpha(t-s)^{\alpha-1}$.

3 The main results

In this section we consider fractional Brownian motion of the Liouville form with Hurst index $H>\frac{1}{2}$ and the following notation is used: C stands for a finite constant not depending on ε and whose value may vary from one occurrence to another.

Theorem 3.1. *Suppose that $\mu(t)$ is an adapted stochastic process and fulfils the following conditions*

$$\sup_{0<t<T}\left(e^{\int_0^t \mu(s)ds}\right) \in L^p(\Omega) \text{ for some } p>\frac{1}{H}, \tag{12}$$

$$e^{\int_0^t \mu(s)ds} \in L^q(\Omega) \text{ for some } q > 4, \tag{13}$$

$$\int_0^T \|\mu(t)\|_{1,4} dt < \infty. \tag{14}$$

Then the equation

$$dX_t = \mu(t)X_t dt + \sigma X_t dW_t^H \ , \ X_0 = x > 0 \tag{15}$$

has a unique solution in **(H)***, which is given by*

$$X_t = X_0 e^{\int_0^t \mu(s)ds + \sigma W_t^H}. \tag{16}$$

Proof. Normally, the way to solve the stochastic differential equations is based on the Itô differential formula. However, in our context this seems impossible since, from Definition 2.1, the solution of (15) is a stochastic process belonging to the space **(H)** and that has a form

$$X_t = X_0 + \int_0^t \mu(s)X_s ds + \int_0^t \sigma X_s K(t,s)\, dW_s + \int_0^t \int_s^t \sigma(X_u - X_s)\, \partial_1 K(u,s) du \delta W_s + \int_0^t \int_0^u \sigma D_s^W X_u \, \partial_1 K(u,s) ds du.$$

In order to find the solution of (15) we consider "approximation" equation driven by semimartingales

$$dX_t^\varepsilon = \mu(t)X_t^\varepsilon dt + \sigma X_t^\varepsilon dW_t^{H,\varepsilon}, X_0^\varepsilon = X_0 = x \tag{17}$$

where $W_t^{H,\varepsilon}$ is defined by (7) with $\varepsilon \in (0,1)$. And then we prove that the limit of X_t^ε in $L^2(\Omega)$ as $\varepsilon \to 0$ will be the solution of (15).

From decomposition (8) above equation can be rewrote into

$$dX_t^\varepsilon = (\mu(t) + \sigma\varphi_t^\varepsilon)X_t^\varepsilon dt + \sigma\varepsilon^\alpha X_t^\varepsilon dW_t. \tag{18}$$

We can see that (18) is a classical Black-Scholes type equation (driven by Brownian motion), so it is easy to find its solution

$$X_t^\varepsilon = X_0 e^{\int_0^t (\mu(s) - \frac{1}{2}\sigma^2\varepsilon^{2\alpha})ds + \sigma W_t^{H,\varepsilon}}. \tag{19}$$

Using chain rule of Malliavin derivative we have for all $u \le t$

$$D_u^W X_t^\varepsilon = \left(\int_u^t D_u^W \mu(s) ds + \sigma(t - u + \varepsilon)^\alpha \right) X_t^\varepsilon \tag{20}$$

Thus $X_t^\varepsilon \in \mathbf{D}_W^{1,2}$ and then by (10) the equation (17) is equivalent to

$$X_t^\varepsilon = X_0 + \int_0^t \mu(s)X_s^\varepsilon ds + \int_0^t \sigma X_s^\varepsilon K(t+\varepsilon,s)\, dW_s + \int_0^t \int_s^t \sigma(X_u^\varepsilon - X_s^\varepsilon)\, \partial_1 K(u+\varepsilon,s) du \delta W_s + \int_0^t \int_0^s \sigma D_u^W X_s^\varepsilon \, \partial_1 K(s+\varepsilon,u) du ds.$$

Hence, in order to prove that X_t defined by (16) is the solution of (15) we need to show that X_t^ε converges to X_t in $L^2(\Omega)$ as $\varepsilon \to 0$ and that both X_t^ε and its limit X_t belong to **(H)**.

Put $Y_t^\varepsilon = \int_0^t (\mu(s) - \frac{1}{2}\sigma^2\varepsilon^{2\alpha})ds + \sigma W_t^{H,\varepsilon}, Y_t = \int_0^t \mu(s)dt + \sigma W_t^H$ then $Y_t^\varepsilon \to Y_t$ in $L^4(\Omega)$ as $\varepsilon \to 0$. Indeed, we have

$$\begin{aligned} E|Y_t^\varepsilon - Y_t|^4 &= E|\tfrac{1}{2}\sigma^2\varepsilon^{2\alpha}t + \sigma(W_t^{H,\varepsilon} - W_t^H)|^4 \\ &\le 8\left((\tfrac{1}{2}\sigma^2\varepsilon^{2\alpha}t)^4 + \sigma^p E|(W_t^{H,\varepsilon} - W_t^H)|^4 \right) \le 8\left((\tfrac{1}{2}\sigma^2\varepsilon^{2\alpha}T)^4 + \sigma^4 c_{4,T}\varepsilon^{4H} \right) \le C\varepsilon^{8\alpha}. \end{aligned} \tag{21}$$

An application of Lagrange's theorem yields

$$E|X_t^\varepsilon - X_t|^2 \le X_0^2 E|A(\varepsilon,t)(Y_t^\varepsilon - Y_t)|^2 \tag{22}$$

where $A(\varepsilon,t) = \sup\limits_{\min(Y_t^\varepsilon, Y_t) \le x \le \max(Y_t^\varepsilon, Y_t)} e^x$ satisfy the following estimate

$$|A(\varepsilon,t)|^2 \le \exp\left(Y_t^\varepsilon + Y_t + |Y_t^\varepsilon - Y_t|\right) \le \frac{ne^{(2+\frac{1}{n})Y_t^\varepsilon} + ne^{(2+\frac{1}{n})Y_t} + e^{(2n+1)|Y_t^\varepsilon - Y_t|}}{2n+1} \; \forall n \ge 1 \; a.s.$$

The condition (13) and Gaussian property of $W_t^{H,\varepsilon}, W_t^H$ imply that $A(\varepsilon,t)$ has finite qth moments (by choosing n large enough). Hence, by applying Hölder inequality we see that there exists a finite constant C not depending on ε such that

$$E|X_t^\varepsilon - X_t|^2 \le C\left(E|Y_t^\varepsilon - Y_t|^4\right)^{\frac{1}{2}} \le C\varepsilon^{4\alpha} \tag{23}$$

which means that X_t^ε converges to X_t in $L^2(\Omega)$ as $\varepsilon \to 0$.

We now are position to prove that X_t^ε belongs to **(H)**. For $0 \le t_1 < t_2 \le T$ and $p \in [2,4]$ we have

$$\begin{aligned} E|X_{t_2}^\varepsilon - X_{t_1}^\varepsilon|^p &\le CX_0^2 E|Y_{t_2}^\varepsilon - Y_{t_1}^\varepsilon|^p \le 2^{p-1}(t_2 - t_1)^{p-1}\int_{t_1}^{t_2} E|\mu(s)|^p ds + 2^{p-1}\sigma^p E|W_{t_2}^{H,\varepsilon} - W_{t_1}^{H,\varepsilon}|^p \\ &\le C((t_2 - t_1)^{p-1} + (t_2 - t_1)^{\frac{p}{2}}) \le C(t_2 - t_1)^{\frac{p}{2}}. \end{aligned} \tag{24}$$

(noting that $E|W_{t_2}^{H,\varepsilon} - W_{t_1}^{H,\varepsilon}|^2 \le C(t_2 - t_1)$, see [15]).

Put $Z_t^\varepsilon = \int_0^t D_u^W \mu(s)ds + \sigma(t - u + \varepsilon)^\alpha$. We have also

$$E|Z_{t_2}^\varepsilon - Z_{t_1}^\varepsilon|^4 \le 8\left((t_2 - t_1)^3 \int_{t_1}^{t_2} E|D_u^W \mu(s)|^4 ds + \sigma^4 (t_2 - t_1)^{4\alpha} \right) \le C(t_2 - t_1)^{4\alpha} \tag{25}$$

(we used a fundamental inequality that $(a+b)^\alpha \le a^\alpha + b^\alpha \; \forall \, a,b > 0, \alpha \in (0,1)$). Consequently

$$\begin{aligned} E|D_u^W X_{t_2}^\varepsilon - D_u^W X_{t_1}^\varepsilon|^2 &\le 2E|Z_{t_2}^\varepsilon (X_{t_2}^\varepsilon - X_{t_1}^\varepsilon)|^2 + 2E|X_{t_1}^\varepsilon (Z_{t_2}^\varepsilon - Z_{t_1}^\varepsilon)|^2 \\ &\le 2\left(E|Z_{t_2}^\varepsilon|^4 E|X_{t_2}^\varepsilon - X_{t_1}^\varepsilon|^4 \right)^{\frac{1}{2}} + 2\left(E|X_{t_1}^\varepsilon|^4 E|Z_{t_2}^\varepsilon - Z_{t_1}^\varepsilon|^4 \right)^{\frac{1}{2}} \le C(t_2 - t_1)^{2\alpha}. \end{aligned} \tag{26}$$

Combining (24) and (26) we get

$$\|X^\varepsilon\|^2_{L_\beta^{1,2}} \le C \sup_{0 < t_1 < t_2 < T} |t_2 - t_1|^{2\alpha - 2\beta}.$$

Thus, X^ε satisfies the condition (i) in **(H)** with any β such that $\frac{1}{2} - H < \beta < \alpha = H - \frac{1}{2}$.

In order to prove X^ε satisfies the condition (ii) in **(H)** we observe that $\sup\limits_{0<t<T}\left(e^{\sigma W_t^{H,\varepsilon}}\right)$ belongs to $L^{p_0}(\Omega)$ for any $p_0 > 1$. Moreover,

$$\sup_{0<t<T} |X_t^\varepsilon| = X_0 \sup_{0<t<T}\left(e^{\int\limits_0^t (\mu(s) - \frac{1}{2}\sigma^2\varepsilon^{2\alpha})ds + \sigma W_t^{H,\varepsilon}}\right) \leq X_0 \sup_{0<t<T}\left(e^{\int\limits_0^t \mu(s)ds}\right) \sup_{0<t<T}\left(e^{\sigma W_t^{H,\varepsilon}}\right). \tag{27}$$

Applying Hölder inequality to (27) and using the condition (12) we get desired result (by choosing p_0 large enough).

The fact X_t belonging to **(H)** can be proved similarly with noting that

$$D_u^W X_t = \left(\int\limits_u^t D_u^W \mu(s)ds + \sigma(t-u)^\alpha\right) X_t.$$

The proof of the theorem is complete.

Remark 3.1. *If we use the Stratonovich integral as in [9] then to the best of our knowledge, there are no any literatures about the explicit solution of the equation (15).*

In order to prove the existence and uniqueness of the solution of (5) we use the method of steps as in the theory of classical delayed differential equations. We shall first prove the result for the interval $[0,r]$, then we use this solution process as the initial condition to solve the equation within the interval $[r,2r]$, and so on. We need the following technical lemma.

Corollary 3.1. *Let μ be a process of bounded from above, i.e. there exists a constant $M > 0$ such that $\mu(t) \leq M$ a.s $\forall t \geq 0$. Moreover,*

$$\int\limits_0^T \|\mu(t)\|_{1,4\times 2^{n_0}} dt < \infty \text{ for some } n_0 \geq 1. \tag{28}$$

Then the unique solution X_t in **(H)** *of the equation (15) satisfies*

$$\int\limits_0^T \|X(t)\|_{1,4\times 2^{n_0-1}} dt < \infty. \tag{29}$$

Proof. The conditions $\mu(t) \leq M$ $a.s.$ and (28) imply (12), (13) and (14). So the equation (15) has a unique solution in **(H)**. The inequality (29) is easy to check since

$$X_t = X_0 e^{\int\limits_0^t \mu(s)ds + \sigma W_t^H} \leq X_0 e^{Mt + \sigma W_t^H},$$

$$D_u^W X_t = \left(\int\limits_u^t D_u^W \mu(s)ds + \sigma(t-u)^\alpha\right) X_t \leq X_0 \left(\int\limits_u^t D_u^W \mu(s)ds + \sigma T^\alpha\right) e^{Mt + \sigma W_t^H}.$$

Theorem 3.2. *The delayed logistic equation driven by fractional Brownian motion (5) admits a unique solution in* **(H)***:*

$$X_t = \phi(t), t \in [-r, 0],$$
$$X_t = \phi(0) e^{\int\limits_0^t (a - bX_{s-r})ds + \sigma W_t^H}, t \in [0,T].$$

Proof. For simplicity let us assume $T = Nr$, where N is a positive integer number. The theorem will be proved by induction where our induction hypothesis, for $n < N$, is the following:

(H_n) The equation

$$X_t = \phi(0) + \int_0^t (a - bX_{s-r})X_s ds + \int_0^t \sigma X_s dW_s^H \ , \ t \in [0, nr], \tag{30}$$

with $X_t = 0\,, t > nr$, has a unique solution in **(H)** *and this solution satisfies*

$$\int_0^T \|X(t)\|_{1,4\times 2^{(N-n)}} dt < \infty. \tag{31}$$

Check (H_1). Let $t \in [0, r]$. Then $X_{t-r} = \phi(t - r)$ and the equation (30) becomes

$$X_t = \phi(0) + \int_0^t \mu_1(s)X_s ds + \int_0^t \sigma X_s dW_s^H \ , \ t \in [0, r], \tag{32}$$

where $\mu_1(s) = a - b\phi(s - r)$.

It is obvious that μ_1 satisfy the conditions in Corollary 3.1 since ϕ is a continuous deterministic function. So (H_1) is true.

Induction. Assume that (H_i) is true for $i \leq n$, with $n < N$. We wish to prove that (H_{n+1}) is true also. Consider the stochastic process defined as

$$V_t = \begin{cases} \phi(t-r) & \text{if } t \leq r, \\ X_{t-r} & \text{if } r < t \leq (n+1)r, \\ 0 & \text{if } t > (n+1)r \end{cases}$$

where X is the solution obtained in (H_n).

Put $\mu_n(s) = a - bV_s$. Thus, for $t \in [0, (n+1)r]$, our problem has become the equation

$$X_t = \phi(0) + \int_0^t \mu_n(s)X_s ds + \int_0^t \sigma X_s dW_s^H \ , \ t \in [0, (n+1)r], \tag{33}$$

Since ϕ is a continuous deterministic function and X_{t-r} is positive a.s. we have μ_n is bounded from above. Moreover, μ_n satisfies the condition (28) with $n_0 = N - n$ by induction hypothesis. Applying Corollary 3.1 one again yields H_{n+1} is true.

The proof of the theorem is complete.

4 Some other models in Physiology

From practical point of view, it is important to find the explicit expression for the solution of each specific model. In this section we want to mention two models in physiology proposed by Mackey and Glass [16] to see more clearly the advantages of semimartingale approximate method. If we assume that the noise is in proportion with the size of population then we get the following two models driven by fractional Brownian motion

$$dx(t) = \left(\lambda - \frac{\alpha V_m x(t) x^n(t-r)}{\theta^n + x^n(t-r)}\right)dt + \sigma x(t) dW_t^H, \tag{34}$$

$$dp(t) = \left(\frac{\beta_0 \theta^n}{\theta^n + p^n(t-r)} - \gamma p(t)\right)dt + \sigma p(t) dW_t^H, \tag{35}$$

where $\lambda, \alpha, V_m, n, r, \theta, \beta_0$ and γ are positive constants.

The first equation is used to study a dynamic disease involving respiratory disorders, where $x(t)$ denotes the arterial CO_2 concentration of a mammal, λ is the CO_2 production rate, V_m, denotes the maximum ”ventilation” rate of CO_2, and r is the time between oxygenation of blood in the lungs and stimulation of chemoreceptors in the brainstem. The last equation is proposed as models of hematopoiesis, $p(t)$ denotes the density of mature cells in blood circulation, and r is the time delay between the production of immature cells in the bone marrow and their maturation for release in the circulating bloodstream.

As we have seen in Section 3, the key problem is to study stochastic differential equation driven by fractional Brownian motion of the following form

$$dX_t = (\lambda(t) + \mu(t)X_t)dt + \sigma X_t dW_t^H \ , \ X_0 = x > 0, \tag{36}$$

where $\mu(t)$ and $\lambda(t)$ are bounded stochastic processes since the size of population is positive and

$\mu(t) = -\frac{\alpha V_m x^n(t-r)}{\theta^n + x^n(t-r)}, \lambda(t) = \lambda$ for equation (34),

$\mu(t) = -\gamma, \lambda(t) = \frac{\beta_0 \theta^n}{\theta^n + p^n(t-r)}$ for equation (35),

Under the bounded condition of $\mu(t)$ and $\lambda(t)$, by using semimartingale approximate method we can find the explicit solution of (36) which is given

$$X_t = \Phi_t \left(X_0 + \int_0^t \lambda(s)(\Phi_s)^{-1} ds \right),$$

where $\Phi_t = \exp\left(\int_0^t \mu(s)ds + \sigma W_t^H\right)$. Then, for example, the solution of (34) is given by

$$x(t) = \exp\left(-\int_0^t \frac{\alpha V_m x^n(s-r)}{\theta^n + x^n(s-r)} ds + \sigma W_t^H \right) \times \left(\phi(0) + \int_0^t \lambda \exp\left(\int_0^s \frac{\alpha V_m x^n(u-r)}{\theta^n + x^n(u-r)} du - \sigma W_s^H \right) ds \right).$$

5 Conclusion

In this paper, we studied the delayed logistic model driven by fractional Brownian motion. Our main contribution is to introduce a method of approximation for finding an explicit expression of the solution. We also showed that this method can be used to study some other models in physiology.

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