A stochastic Ginzburg-Landau equation with impulsive effects

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Abstract

In this paper we consider a stochastic Ginzburg-Landau equation with impulsive effects. We first prove the existence and uniqueness of the global solution which can be explicitly represented via the solution of a stochastic equation without impulses. Then, based on our obtained result, we study the qualitative properties of the solution, including the boundedness of moments, almost surely exponential convergence and pathwise estimations. Finally, we give a first attempt to study a fractional version of impulsive stochastic Ginzburg-Landau equations.

Keywords: Stochastic Ginzburg-Landau equation, impulses, asymptotic behavior.

1. Introduction

The deterministic Ginzburg-Landau equation was introduced by Ginzburg and Landau (1950) in [1] to describe a phase transition in the theory of superconductivity. Since then, it has appeared in many different contexts such as nonlinear optics with dissipation, the theory of bistable systems, etc. In addition, it plays an important role as modulation equation and it serves as a simple mathematical model for studying the transition from regular to turbulent behavior [2].

In the last decades, a lot of stochastic versions of Ginzburg-Landau equations have been introduced and studied by many authors. For example, Kloeden and Platen (1992) [3] provided an explicit solution to stochastic Ginzburg-Landau equation with multiplicative noise:

$$dX_t = \big((a + \frac{\sigma^2}{2})X_t - bX_t^3\big)dt + \sigma X_t dW_t\,, \tag{1.1}$$

where W_t is a standard Brownian motion, a, σ and $b > 0$ are constants.

Neiman and Geier (1994) in [4] studied stochastic resonance in an overdamped bistable system driven by white and harmonic noises in the form

$$\frac{dx(t)}{dt} = x - x^3 + \sqrt{2D}\xi(t) + y(t),$$

where $\xi(t)$ is Gaussian white noise and $y(t)$ is harmonic noise which is independent from $\xi(t)$.

Tsimring and Pikovsky (2001) in [5] and Goulding et al. (2007) in [6] considered the equations with time delay:

$$\frac{dx(t)}{dt} = x(t) - x(t)^3 + \varepsilon x(t-\tau) + \sqrt{2D}\xi(t),$$

where $\tau > 0$ is the time delay and ε is the strength of the feedback. The solution $x(t)$ describes the stochastic evolution of the position of a particle trapped in a double well potential $U(x) = \frac{x^4}{4} - \frac{x^2}{2}$ in the presence of a time delayed force $\varepsilon x(t-\tau)$ and of Gaussian white noise $\xi(t)$.

Brassesco et al. (1995) in [7] and Fatkullin & Vanden-Eijnden (2002) in [8] investigated the stochastic Ginzburg-Landau equation of the following form

$$\frac{\partial m}{\partial t}(t,x) = \frac{1}{2}\frac{\partial^2 m}{\partial x^2}(t,x) + [m(t,x) - m(t,x)^3] + \sqrt{\varepsilon}\xi(x,t),$$

where $\xi(x,t)$ is a white noise in space and time. This equation appears in the literature as a model for phase separation and interface dynamics in systems with non conserved order parameter. Its solution also describes the spatio-temporal evolution of a bistable system.

On the other hand, it is known that the impulsive effects exist widely in the different areas of real world such as mechanics, electronics, telecommunications, neural networks, finance and economics, etc. This is due to the fact that the states of many evolutionary processes are often subject to instantaneous perturbations and experience abrupt changes at certain moments of time. The duration of these changes is very short and negligible in comparison with the duration of the process considered, and can be thought as impulses. Naturally, systems with short-term perturbations should be described by impulsive differential equations and in fact, the theory of impulsive differential equations has been studied extensively. For more details, we refer the reader to [9, 10, 11] for deterministic theory and to [12, 13, 14] for the case of stochastic one.

From the above discussions, it is of great significance to take into account the effect of impulses in the investigation of stochastic Ginzburg-Landau equations. However, to the best of our knowledge, the results about stochastic Ginzburg-Landau equations with impulsive effects are scarce.

In this paper, inspired by model (1.1), we study the following stochastic Ginzburg-Landau equation with impulsive effects

$$\begin{cases} dX_t = \big(a(t)X_t - b(t)X_t^3\big)dt + \sigma(t)X_t dW_t, \;\; t \neq t_k, \;\; k \in \mathbb{N}, \\ X_{t_k^+} - X_{t_k} = \lambda_k X_{t_k}, \;\; k \in \mathbb{N}, \end{cases} \tag{1.2}$$

with the initial condition $X_0 > 0$, where $\mathbb{N}$ denotes the set of positive integers, $0 < t_1 < t_2 < \ldots..$, $\lim_{k\to\infty} t_k = \infty$, $a(t), b(t)$ and $\sigma(t)$ are bounded continuous functions on $\mathbb{R}_+ = [0, \infty)$. In addition, we assume that

$$\underline{b} := \inf_{t\in\mathbb{R}_+} b(t) > 0.$$

It is known that the traditional tools to study stochastic differential equation (such as Itô formula) cannot be effectively used for impulsive stochastic differential equations, since it is difficult to deal with when integrating intervals contain impulses. In order to avoid this difficulty we will point out the relation between the solution of impulsive equation (1.2) and the solution of a corresponding equation without impulses. Thus the traditional methods can be applied.

This paper is organized as follows. In Section 2, we give an explicit expression for the solution and show the boundedness of moments. Section 3 is devoted to studying some qualitative properties of the solution, including almost surely exponentially convergent and long term asymptotic behaviors. Section 4 contains some comments on a fractional stochastic version with impulses. The conclusion is given in Section 5.

2. The unique global solution and its representation

Throughout this paper, we use the following notations. Denote $\mathbb{R}_+ = [0, \infty)$. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, that is, it is right continuous and increasing while $\mathcal{F}_0$ contains all P-null sets. Let W_t be a standard Brownian motion defined on this probability space and consider the impulsive stochastic Ginzburg-Landau equation (1.2).

For g, a bounded continuous function on $\mathbb{R}_+$, we denote

$$\underline{g} = \inf_{t\in\mathbb{R}_+} g(t), \;\; \bar{g} = \sup_{t\in\mathbb{R}_+} g(t).$$

Definition 2.1. The solution of the Ginzburg-Landau equation (1.2) is a stochastic process $\{X_t, t \in \mathbb{R}_+\}$ such that

(i) X_t is $\mathcal{F}_t$-adapted and is continuous on $(0, t_1]$ and each interval $(t_k, t_{k+1}], k \in \mathbb{N}$,

(ii) for each t_k, $X_{t_k^+} = \lim\limits_{t \to t_k^+} X_t$ exists and $X_{t_k^+} - X_{t_k} = \lambda_k X_{t_k}$ with probability one,

(iii) X_t satisfies the following integral equations a.s.

$$X_t = X_0 + \int_0^t \big(a(s)X_s - b(s)X_s^3\big)ds + \int_0^t \sigma(s)X_s dW_s, \;\; t \in [0, t_1],$$

$$X_t = X_{t_k^+} + \int_{t_k}^t \big(a(s)X_s - b(s)X_s^3\big)ds + \int_{t_k}^t \sigma(s)X_s dW_s, \;\; t \in (t_k, t_{k+1}], k \in \mathbb{N},$$

provided that the integrals exist.

In the Theorems below we always assume that a product equals unity if the number of factors is zero.

Theorem 2.1. *The Ginzburg-Landau equation (1.2) admits a unique solution which is defined globally and given by*

$$X_t = \frac{\prod\limits_{0<t_k<t} (1+\lambda_k) e^{\int_0^t [a(s) - \frac{1}{2}\sigma^2(s)]ds + \int_0^t \sigma(s)dW_s}}{\left(X_0^{-2} + 2\int_0^t b(s) \prod\limits_{0<t_k<s} (1+\lambda_k)^2 e^{2\int_0^s [a(u)-\frac{1}{2}\sigma^2(u)]du + 2\int_0^s \sigma(u)dW_u} ds\right)^{\frac{1}{2}}}. \tag{2.1}$$

Proof. We first define the stochastic process

$$\begin{aligned} U_t = {} & e^{-2\int_0^t [a(s)-\frac{1}{2}\sigma^2(s)]ds - 2\int_0^t \sigma(s)dW_s} \\ & \times \left(X_0^{-2} + 2\int_0^t \prod_{0<t_k<s} (1+\lambda_k)^2 b(s) e^{2\int_0^s [a(u)-\frac{1}{2}\sigma^2(u)]du + 2\int_0^s \sigma(u)dW_u} ds\right) := e^{V_t} \times f(t). \end{aligned} \tag{2.2}$$

By the Itô formula, U_t solves

$$dU_t = f'(t)e^{V_t}dt + f(t)e^{V_t}dV_t + \frac{1}{2}f(t)e^{V_t}(dV_t)^2,$$

or equivalently

$$\begin{aligned} dU_t &= 2\prod_{0<t_k<t}(1+\lambda_k)^2 b(t)dt - 2U_t[a(t)-\tfrac{1}{2}\sigma^2(t)]dt - 2U_t\sigma(t)dW_t + 2U_t\sigma^2(t)dt \\ &= [3\sigma^2(t)-2a(t)]U_t dt - 2\sigma(t)U_t dW_t + 2\prod_{0<t_k<t}(1+\lambda_k)^2 b(t)dt. \end{aligned} \tag{2.3}$$

Now let $Y_t = \frac{1}{\sqrt{U_t}}$, then

$$\begin{aligned} dY_t &= \tfrac{-1}{2U_t\sqrt{U_t}}dU_t + \tfrac{3}{8U_t^2\sqrt{U_t}}(dU_t)^2 \\ &= \tfrac{-1}{2}[3\sigma^2(t)-2a(t)]Y_t dt + \sigma(t)Y_t dW_t - \prod_{0<t_k<t}(1+\lambda_k)^2 b(t)Y_t^3 dt + \tfrac{3}{2}\sigma^2(t)Y_t dt. \end{aligned}$$

Thus Y_t is the solution of the following equation without impulses

$$dY_t = \left(a(t)Y_t - b(t)\prod_{0<t_k<t}(1+\lambda_k)^2 Y_t^3\right)dt + \sigma(t)Y_t dW_t, \;\; Y_0 = X_0. \tag{2.4}$$

Consider the stochastic process

$$\begin{aligned} X_t &= \prod_{0<t_k<t}(1+\lambda_k)Y_t \\ &= \frac{\prod_{0<t_k<t}(1+\lambda_k)e^{\int_0^t[a(s)-\frac{1}{2}\sigma^2(s)]ds+\int_0^t\sigma(s)dW_s}}{\left(X_0^{-2}+2\int_0^t\prod_{0<t_k<s}(1+\lambda_k)^2 b(s)e^{2\int_0^s[a(u)-\frac{1}{2}\sigma^2(u)]du+2\int_0^s\sigma(u)dW_u}ds\right)^{\frac{1}{2}}}. \end{aligned} \tag{2.5}$$

Obviously, X_t is $\mathcal{F}_t$-adapted and is continuous on each interval $(0,t_1]$ and $(t_k,t_{k+1}], k\in\mathbb{N}$. On the other hand, for each t_k

$$\begin{aligned} X_{t_k^+} &= \lim_{t\to t_k^+} X_t = \lim_{t\to t_k^+}\prod_{0<t_h<t}(1+\lambda_h)Y_t \\ &= \prod_{0<t_h\le t_k}(1+\lambda_h)Y_{t_k^+} = (1+\lambda_k)\prod_{0<t_h<t_k}(1+\lambda_h)Y_{t_k} = (1+\lambda_k)X_{t_k}, \end{aligned}$$

which means that $X_{t_k^+}$ exists and X_t satisfies the impulsive conditions at each $t_k, k\in\mathbb{N}$. Moreover, for any $t\neq t_k$ we have

$$\begin{aligned} dX_t &= \prod_{0<t_k<t}(1+\lambda_k)dY_t \\ &= \left(a(t)\prod_{0<t_k<t}(1+\lambda_k)Y_t - b(t)\prod_{0<t_k<t}(1+\lambda_k)^3 Y_t^3\right)dt + \sigma(t)\prod_{0<t_k<t}(1+\lambda_k)Y_t dW_t \\ &= \big(a(t)X_t - b(t)X_t^3\big)dt + \sigma(t)X_t dW_t. \end{aligned}$$

This equation says that X_t satisfies the integral equations appearing in the condition (iii) of Definition 2.1. The remainder of the proof is to show that X_t is the unique solution. Noting that on each interval $[0, t_1]$ and $(t_k, t_{k+1}], k \in \mathbb{N}$, the system (1.2) a standard Itô stochastic differential equation. Since its coefficients are local Lipschitz continuous, the uniqueness of the solution is clear (see, for instance, [15]).

The Theorem is proved. $\square$

Remark 2.1. We observe from (2.1) that the solution of impulsive Ginzburg-Landau equation (1.2) can be negative, depending on the sign of $(1+\lambda_k)'s$. In particular, denote by t_k the first moment of time such that $\lambda_k = -1$. Our system will vanish right after this moment, i.e., $X_t = 0$ for all $t > t_k$.

On the other hand, the solution of the original Ginzburg-Landau equation, that is system (1.2) without impulses, is given by

$$X_t = \frac{e^{\int\limits_0^t [a(s)-\frac{1}{2}\sigma^2(s)]ds + \int\limits_0^t \sigma(s)dW_s}}{\left(X_0^{-2} + 2\int\limits_0^t b(s) e^{2\int\limits_0^s [a(u)-\frac{1}{2}\sigma^2(u)]du + 2\int\limits_0^s \sigma(u)dW_u} ds\right)^{\frac{1}{2}}}. \tag{2.6}$$

This solution is globally positive for any initial value $X_0 > 0$. This means that the dynamic of impulsive model is very different from the one of its original model.

In the remainder of the paper, it is more interesting to consider the case where $\lambda_k \neq -1$, *i.e.* $(1+\lambda_k)^2 > 0$ for all $k \in \mathbb{N}$.

Theorem 2.2. *Suppose that there exist two positive constants m, M such that*

$$m \leq \prod_{0<t_k<t} (1+\lambda_k)^2 \leq M \;\forall\, t \in \mathbb{R}_+. \tag{2.7}$$

Then for each $p > 0$, there exists a finite positive constant $C_p(m, X_0)$ such that

$$\sup_{t\in\mathbb{R}_+} E|X_t|^p \leq C_p(m, X_0) M^{\frac{p}{2}}.$$

Proof. By Itô formula

$$dY_t^p = \left(pa(t)Y_t^p + \frac{1}{2}p(p-1)\sigma^2(t)Y_t^p - p\prod_{0<t_k<t}(1+\lambda_k)^2 b(t)Y_t^{p+2}\right)dt + p\sigma(t)Y_t^p dW_t.$$

Hence, by Lyapunov's inequality $(E[Y_t^p])^{1+\frac{1}{p}} \le (E[Y_t^{p+2}])^{1+\frac{2}{p}}$ we have

$$\begin{aligned} E[Y_t^p] &= Y_0^p + p\int_0^t \Big([a(s)+\tfrac{1}{2}(p-1)\sigma^2(s)]E[Y_s^p] - \prod_{0<t_k<s}(1+\lambda_k)^2 b(s)E[Y_s^{p+2}]\Big)ds \\ &\le Y_0^p + p\int_0^t \Big([\bar{a}+\tfrac{1}{2}(p-1)\bar{\sigma}^2]E[Y_t^p] - m\underline{b}(E[Y_t^p])^{1+\frac{2}{p}}\Big)dt. \end{aligned} \tag{2.8}$$

It follows from the differential inequalities (see, [16]) that $E[Y_t^p]$ is dominated by the solution of the following ordinary Bernoulli equation

$$\frac{dy}{dt} = P(t)y^n + Q(t)y, \;\; y(0) = Y_0^p, \tag{2.9}$$

where $P(t) = -pm\underline{b}, Q(t) = p[\bar{a}+\frac{1}{2}(p-1)\bar{\sigma}^2]$. Solving (2.9) gives us

$$E[Y_t^p] \le y(t) = e^{p[\bar{a}+\frac{1}{2}(p-1)\bar{\sigma}^2]t}\left(Y_0^{-2} + 2m\underline{b}\int_0^t e^{2[\bar{a}+\frac{1}{2}(p-1)\bar{\sigma}^2]s}ds\right)^{\frac{-p}{2}}.$$

Now we choose $p_0 > 0$ such that $\bar{a}+\frac{1}{2}(p_0-1)\bar{\sigma}^2 > 0$. An easy computation leads us the estimate for any $p \ge p_0$

$$\begin{aligned} (E[Y_t^p])^{\frac{2}{p}} &\le \frac{e^{2[\bar{a}+\frac{1}{2}(p-1)\bar{\sigma}^2]t}}{Y_0^{-2} + 2m\underline{b}\int_0^t e^{2[\bar{a}+\frac{1}{2}(p-1)\bar{\sigma}^2]s}ds} \\ &= \frac{1}{\left(\frac{1}{X_0^2} - \frac{m\underline{b}}{\bar{a}+\frac{1}{2}(p-1)\bar{\sigma}^2}\right)e^{-2[\bar{a}+\frac{1}{2}(p-1)\bar{\sigma}^2]t} + \frac{m\underline{b}}{\bar{a}+\frac{1}{2}(p-1)\bar{\sigma}^2}} \\ &\le \max\{\frac{m\underline{b}}{\bar{a}+\frac{1}{2}(p-1)\bar{\sigma}^2}, X_0^2\}. \end{aligned}$$

We therefore have for any $p \ge p_0$

$$E[Y_t^p] \le \max\{\frac{m\underline{b}}{\bar{a}+\frac{1}{2}(p-1)\bar{\sigma}^2}, X_0^2\}^{\frac{p}{2}} < \infty \;\forall\, t \in \mathbb{R}_+.$$

For $p < p < p_0$, by Lyapunov's inequality we also have $E[Y_t^p] \le (E[Y_t^{p_0}])^{\frac{p}{p_0}} < \infty$. Thus for each $p > 0$, there exists a finite positive constant $C_p(m, X_0)$ such that $E[Y_t^p] \le C_p(m, X_0)$ and

$$E|X_t|^p = \prod_{0<t_k<t}|1+\lambda_k|^p E[Y_t^p] \le C_p(m, X_0)M^{\frac{p}{2}} \;\forall\, t \in \mathbb{R}_+.$$

The Theorem is proved. □

Remark 2.2. In the general case, we always have the following estimate

$$E|X_t|^p \le \frac{\prod\limits_{0<t_k<t} |1+\lambda_k|^p e^{p\int\limits_0^t [a(s)+\frac{1}{2}(p-1)\sigma^2(s)]ds}}{\left(X_0^{-2} + 2\int\limits_0^t \prod\limits_{0<t_k<s} (1+\lambda_k)^2 b(s) e^{2\int\limits_0^s [a(u)+\frac{1}{2}(p-1)\sigma^2(u)]du} ds\right)^{\frac{p}{2}}}.$$

3. Asymptotic behavior of solutions

In this section we shall investigate the some qualitative properties of the solution of (1.2) which allow us to gain a deeper understanding about its dynamics.

Theorem 3.1. *Assume that*

$$\limsup_{t\to\infty} \frac{1}{t}\Bigg(\sum_{0<t_k<t} \ln|1+\lambda_k| + \int\limits_0^t [a(s) - \frac{1}{2}\sigma^2(s)]ds \Bigg) < 0 \ a.s.$$

Then the solution of impulsive Ginzburg-Landau equation (1.2) is almost surely exponentially convergent, i.e. there exists $\beta > 0$ such that

$$\limsup_{t\to\infty} \frac{\ln|X_t|}{t} \le -\beta \quad a.s. \tag{3.1}$$

Proof. Using Itô formula, it is easy to get that

$$d\ln Y_t = \Bigg(a(t) - \frac{1}{2}\sigma^2(t) - \prod_{0<t_k<t} (1+\lambda_k)^2 b(t) Y_t^2\Bigg)dt + \sigma(t)dW_t.$$

Consequently,

$$\ln Y_t - \ln Y_0 \le \int\limits_0^t [a(s) - \frac{1}{2}\sigma^2(s)]ds + \int\limits_0^t \sigma(s)dW_s,$$

and hence, it follows from the relation (2.5) that

$$\ln|X_t| - \ln X_0 \le \sum_{0<t_k<t} \ln|1+\lambda_k| + \int\limits_0^t [a(s) - \frac{1}{2}\sigma^2(s)]ds + \int\limits_0^t \sigma(s)dW_s. \tag{3.2}$$

Put $M_t = \int\limits_0^t \sigma(s)dW_s$. Then M_t is a martingale of finite quadratic variation:

$$\langle M, M\rangle_t = \int\limits_0^t \sigma^2(s)ds \leq \bar{\sigma}^2 t.$$

Using the strong law of large numbers for martingales [17, Theorem 3.4] we have

$$\lim_{t\to\infty} \frac{M_t}{t} = 0 \ a.s. \tag{3.3}$$

Combining (3.2) and (3.3) we get (3.1) with

$$\beta = -\limsup_{t\to\infty} \frac{1}{t}\bigg(\sum_{0<t_k<t} \ln|1+\lambda_k| + \int\limits_0^t [a(s) - \frac{1}{2}\sigma^2(s)]ds\bigg). \tag{3.4}$$

The Theorem is proved. $\square$

Theorem 3.2. *Under the assumption of Theorem 2.2. We have*

$$\limsup_{t\to\infty} \frac{\ln|X_t|}{\ln t} \leq 1 \ \ a.s. \tag{3.5}$$

Proof. Applying the Itô formula to $Z_t = \ln Y_t$ and then to $e^{nt}Z_t$ we get, respectively

$$dZ_t = \bigg(a(t) - \frac{1}{2}\sigma^2(t) - \prod_{0<t_k<t}(1+\lambda_k)^2 b(t)e^{2Z_t}\bigg)dt + \sigma(t)dW_t,$$

$$d(e^{nt}Z_t) = ne^{nt}Z_t dt + e^{nt}\bigg(a(t) - \frac{1}{2}\sigma^2(t) - \prod_{0<t_k<t}(1+\lambda_k)^2 b(t)e^{2Z_t}\bigg)dt + e^{nt}\sigma(t)dW_t.$$

Therefore

$$e^{nt}Z_t = Z_0 + \int\limits_0^t e^{ns}\bigg(a(s) - \frac{1}{2}\sigma^2(s) + nZ_s - \prod_{0<t_k<s}(1+\lambda_k)^2 b(s)e^{2Z_s}\bigg)ds + \int\limits_0^t e^{ns}\sigma(s)dW_s.$$

Since $\prod\limits_{0<t_k<s}(1+\lambda_k)^2 b(s) \geq m\bar{b} > 0$ for all s and $a(s), \sigma(s)$ are bounded functions, there exists a finite positive K such that for any $s \in \mathbb{R}_+, Z \in \mathbb{R}$

$$a(s) - \frac{1}{2}\sigma^2(s) + nZ_s - \prod_{0<t_k<s}(1+\lambda_k)^2 b(s)e^{2Z_s} \leq K.$$

Consequently,

$$e^{nt}Z_t \le Z_0 + K\int_0^t e^{ns}ds + \int_0^t e^{ns}\sigma(s)dW_s. \tag{3.6}$$

Put

$$M_t = \int_0^t \sigma_1 e^{ns}dW_s,$$

then M_t is continuous martingale that has finite quadratic variation:

$$\langle M, M\rangle_t = \int_0^t \sigma^2(s)e^{2ns}ds.$$

Fix $\varepsilon \in (0,1)$ and $\theta > 1$, by applying the exponential martingale inequality (see, [17, Theorem 7.4]) we have for any $k \ge 1$

$$P\left(\sup_{0\le t\le k}\left(M_t - \frac{\varepsilon}{2}e^{-nk}\langle M, M\rangle_t\right) \ge \frac{e^{nk}\ln k^\theta}{\varepsilon}\right) \le \frac{1}{k^\theta}.$$

Since $\sum\limits_{k=1}^{\infty}\frac{1}{k^\theta} < \infty$, an application of the Borel-Cantelli lemma yields that there exist $\Omega_1 \subset \Omega$ with $P(\Omega_1) = 1$ such that for any $\omega \in \Omega_1$ there exists an integer $k(\omega)$, when $k \ge k(\omega)$ and $k-1 \le t \le k$,

$$\begin{aligned} M_t &\le \tfrac{\varepsilon}{2}e^{-nk}\langle M, M\rangle_t + \tfrac{e^{nk}\ln k^\theta}{\varepsilon} \\ &= \tfrac{\varepsilon}{2}e^{-nk}\int_0^t \sigma^2(s)e^{2ns}ds + \theta\tfrac{e^{nk}\ln k}{\varepsilon}. \end{aligned}$$

Substituting this inequality into (3.6) results in

$$e^{nt}Z_t \le Z_0 + \frac{K}{n}e^{nt} + \frac{\varepsilon}{2}e^{-nk}\int_0^t \sigma^2(s)e^{2ns}ds + \theta\frac{e^{nk}\ln k}{\varepsilon},\ k-1 \le t \le k,$$

$$\begin{aligned} Z_t &\le Z_0 + \tfrac{K}{n} + \tfrac{\varepsilon\bar{\sigma}^2}{2}e^{-n(k-t)} + \theta\tfrac{e^{n(k-t)}\ln k}{\varepsilon} \\ &\le Z_0 + \tfrac{K}{n} + \tfrac{\varepsilon\bar{\sigma}^2}{2} + \theta\tfrac{e^{n}\ln k}{\varepsilon},\ k-1 \le t \le k. \end{aligned} \tag{3.7}$$

which implies that

$$\limsup_{t\to\infty}\frac{Z_t}{\ln t} \le \theta\frac{e^n}{\varepsilon}.$$

Taking the limits $\theta \to 1^+, \varepsilon \to 1^-$ and $n \to 0^+$ we get

$$\limsup_{t\to\infty} \frac{\ln Y_t}{\ln t} = \limsup_{t\to\infty} \frac{Z_t}{\ln t} \leq 1 \ a.s. \tag{3.8}$$

It follows from (2.7) that

$$\lim_{t\to\infty} \frac{\sum\limits_{0<t_k<t} \ln|1+\lambda_k|}{\ln t} = 0.$$

As a consequence,

$$\limsup_{t\to\infty} \frac{\ln|X_t|}{\ln t} = \limsup_{t\to\infty} \frac{\sum\limits_{0<t_k<t} \ln|1+\lambda_k| + \ln Y_t}{\ln t} \leq 1 \ a.s.$$

We finish the proof of Theorem. $\square$

Theorem 3.3. *Under the assumption of Theorem 2.2, we additionally assume that* $\underline{c} := \inf\limits_{t\in\mathbb{R}_+} [a(t) - \frac{1}{2}\sigma^2(t)] > 0$, *then*

$$\liminf_{t\to\infty} \frac{\ln|X_t|}{\ln t} \geq -\frac{\bar{\sigma}^2}{2\underline{c}} \quad a.s. \tag{3.9}$$

Proof. Since $\underline{c} > 0$, we can choose a positive constant θ such that $\underline{c} > \theta\bar{\sigma}^2$. Consider the Lyapunov functional $V(y) = (1 + \frac{1}{y^2})^\theta$ and applying the Itô formula to $V(Y_t)$

$$dV(Y_t) = d(1+U_t)^\theta = \theta(1+U_t)^{\theta-1} dU_t + \frac{1}{2}\theta(\theta-1)(1+U_t)^{\theta-2}(dU_t)^2,$$

where U_t is defined by the equation (2.3). Hence,

$$\begin{aligned} dV(Y_t) = \theta(1+U_t)^{\theta-2}\Big(&[3\sigma^2(t) - 2a(t)]U_t(1+U_t)dt - 2\sigma(t)U_t(1+U_t)dW_t \\ &+ 2\prod_{0<t_k<t}(1+\lambda_k)^2 b(t)(1+U_t)dt + 2(\theta-1)\sigma^2(t)U_t^2 dt\Big), \end{aligned}$$

and then

$$\begin{aligned} dV(Y_t) &= \theta(1+U_t)^{\theta-2}\Big(-2U_t^2[a(t) - \frac{1}{2}\sigma^2(t) - \theta\sigma^2(t)] + U_t[3\sigma^2(t) - 2a(t) \\ &\quad + 2\prod_{0<t_k<t}(1+\lambda_k)^2 b(t)] + 2\prod_{0<t_k<t}(1+\lambda_k)^2 b(t)\Big)dt - 2\theta\sigma(t)U_t(1+U_t)^{\theta-1}dW_t \\ &\leq \theta(1+U_t)^{\theta-2}\big(-2[\underline{c} - \theta\bar{\sigma}^2]U_t^2 + [3\bar{\sigma}^2 + 2M\bar{b}]U_t + 2M\bar{b}\big)dt \\ &\quad - 2\theta\sigma(t)U_t(1+U_t)^{\theta-1}dW_t. \end{aligned} \tag{3.10}$$

Once again, we apply the Itô formula to $e^{nt}V(Y_t)$ to get

$$\begin{aligned}
d(e^{nt}V(Y_t)) &= ne^{nt}V(Y_t)dt + e^{nt}dV(Y_t)\\
&\le \theta(1+U_t)^{\theta-2}\big(\frac{n}{\theta}(1+U_t)^2 - 2[\underline{c}-\theta\bar{\sigma}^2]U_t^2 + [3\bar{\sigma}^2+2M\bar{b}]U_t + 2M\bar{b}\big)e^{nt}dt\\
&\qquad - 2\theta\sigma(t)e^{nt}U_t(1+U_t)^{\theta-1}dW_t\\
&= \theta(1+U_t)^{\theta-2}\big(-2[\underline{c}-\theta\bar{\sigma}^2-\frac{n}{2\theta}]U_t^2 + [3\bar{\sigma}^2+2M\bar{b}+\frac{2n}{\theta}]U_t + 2M\bar{b}+\frac{n}{\theta}\big)dt\\
&\qquad - 2\theta\sigma(t)e^{nt}U_t(1+U_t)^{\theta-1}dW_t. \quad (3.11)
\end{aligned}$$

Now we choose $n>0$ such that $\underline{c}-\theta\bar{\sigma}^2-\frac{n}{2\theta}>0$, it is very easy to check that the function

$$g(u) := \theta(1+u)^{\theta-2}\big(-2[\underline{c}-\theta\bar{\sigma}^2-\frac{n}{2\theta}]u^2 + [3\bar{\sigma}^2+2M\bar{b}+\frac{2n}{\theta}]u + 2M\bar{b}+\frac{n}{\theta}\big),\ u>0$$

is bounded by a finite positive constant, namely K. Consequently,

$$d(e^{nt}V(Y_t)) \le Ke^{nt}dt - 2\theta\sigma(t)e^{nt}U_t(1+U_t)^{\theta-1}dW_t,$$

which implies that

$$E[e^{nt}V(Y_t)] \le V(Y_0) + \frac{K}{n}e^{nt}\ \forall\ t\in\mathbb{R}_+,$$

and so

$$E[V(Y_t)] \le V(Y_0) + \frac{K}{n} := K_1\ \forall\ t\in\mathbb{R}_+. \quad (3.12)$$

From (3.10) we obtain

$$dV(Y_t) \le K_2\theta(1+U_t)^{\theta} - 2\theta\sigma(t)U_t(1+U_t)^{\theta-1}dW_t,$$

where $K_2=\max\{2[\underline{c}-\theta\bar{\sigma}^2], 3\bar{\sigma}^2+2M\bar{b}, 2M\bar{b}\}$. Let $k=1,2,....$ and $v>0$, the latest inequality leads us to the following estimate

$$\begin{aligned}
E\Big(\sup_{(k-1)v\le t\le kv} V(Y_t)\Big) &\le EV(Y_{(k-1)v}) + E\Big(\sup_{(k-1)v\le t\le kv}\int_{(k-1)v}^{t} K_2\theta V(Y_s)ds\Big)\\
&\quad + E\Big(\sup_{(k-1)v\le t\le kv}\int_{(k-1)v}^{t} 2\theta\sigma(s)U_s(1+U_s)^{\theta-1}dW_s\Big). \quad (3.13)
\end{aligned}$$

Clearly,

$$E\Big(\sup_{(k-1)v\le t\le kv}\int_{(k-1)v}^{t} K_2\theta V(Y_s)ds\Big) \le K_2\theta vE\Big(\sup_{(k-1)v\le t\le kv} V(Y_t)\Big). \quad (3.14)$$

By the Burkholder-Davis-Gundy inequality

$$E\Big(\sup_{(k-1)v\le t\le kv}\int\limits_{(k-1)v}^{t} 2\theta\sigma(s)U_s(1+U_s)^{\theta-1}dW_s\Big)$$

$$\le 2E\Big(\int\limits_{(k-1)v}^{kv}\theta^2\sigma^2(s)U_s^2(1+U_s)^{2\theta-2}ds\Big)^{\frac{1}{2}}$$

$$\le 2\theta\bar{\sigma}E\Big(\int\limits_{(k-1)v}^{kv}(1+U_s)^{2\theta}ds\Big)^{\frac{1}{2}}$$

$$\le 2\theta\bar{\sigma}v^{\frac{1}{2}}E\Big(\sup_{(k-1)v\le t\le kv}V(Y_t)\Big). \quad (3.15)$$

Inserting (3.14) and (3.15) into (3.13) and using (3.12) we obtain

$$E\Big(\sup_{(k-1)v\le t\le kv}V(Y_t)\Big)\le K_1+(K_2\theta v+2\theta\bar{\sigma}v^{\frac{1}{2}})E\Big(\sup_{(k-1)v\le t\le kv}V(Y_t)\Big). \quad (3.16)$$

Now we choose $v>0$ such that $K_2\theta v+2\theta\bar{\sigma}v^{\frac{1}{2}}\le\frac{1}{2}$, then

$$E\Big(\sup_{(k-1)v\le t\le kv}V(Y_t)\Big)\le 2K_1. \quad (3.17)$$

Fix $\varepsilon>1$, by the Chebyshev inequality

$$P\Big(\sup_{(k-1)v\le t\le kv}V(Y_t)>(kv)^{\varepsilon}\Big)\le\frac{E\Big(\sup_{(k-1)v\le t\le kv}V(Y_t)\Big)}{(kv)^{\varepsilon}}\le\frac{2K_1}{(kv)^{\varepsilon}},\ k=1,2,....$$

Making use of the Borel-Cantelli lemma yields that there exists an integer $k(\omega)$, when $k\ge k(\omega)$ and $(k-1)v\le t\le kv$,

$$V(Y_t)\le(kv)^{\varepsilon}\ a.s.$$

Similarly to (3.8) we also have

$$\limsup_{t\to\infty}\frac{\ln V(Y_t)}{\ln t}\le 1\ a.s.$$

Since $V(Y_t)=\big(1+\frac{1}{Y_t^2}\big)^{\theta}$, this implies that

$$\limsup_{t\to\infty}\frac{\ln(Y_t)^{-2\theta}}{\ln t}\le 1\ a.s.$$

Thus

$$\liminf_{t\to\infty} \frac{\ln Y_t}{\ln t} \geq -\frac{1}{2\theta} \ a.s.$$

The latest inequality holds for any θ such that $\underline{c} > \theta\bar{\sigma}^2$. We conclude that

$$\liminf_{t\to\infty} \frac{\ln Y_t}{\ln t} \geq -\frac{\bar{\sigma}^2}{2\underline{c}} \quad a.s.$$

and so (3.9) is proved.

The proof of the theorem is completed. □

4. A stochastic version driven by fractional Brownian motion

In the last two decades, there has been an increased interest in stochastic models based on other processes rather than the Brownian motion, much of the literature has pointed out that fractional Brownian motion, also well known as colored noise, provides a natural theoretical framework to model many phenomena arising in finance, biology, physics, etc. We refer also the reader to [18] for a short survey on the existence of colored noise in the real world. This naturally leads us to investigate impulsive stochastic differential equations driven by fractional Brownian motion. However, to the best of the author's knowledge, this field has not yet been established even in the simplest case. The aim of this section is to consider an impulsive stochastic Ginzburg-Landau equation driven by fractional Brownian motion.

The fractional Brownian motion (fBm) of the Hurst parameter $H \in (0,1)$ is a centered Gaussian process $W^H = \{W^H(t), t \geq 0\}$ with the covariance function $R_H(t,s) = E[W_t^H W_s^H]$

$$R_H(t,s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}).$$

It is known that fBm is a self-similar process and has stationary increments. In the case where $H = \frac{1}{2}$, the process W^H reduces to a standard Brownian motion. The increments of the fBm are negatively correlated for $H < \frac{1}{2}$ and positively correlated for $H > \frac{1}{2}$. In particular, for $H > \frac{1}{2}$, fBm is a long memory process since the covariance at distance n decreases as n^{2H-2} :

$$\rho_H(n) := E(W_1^H(W_{n+1}^H - W_n^H)) \approx H(2H-1)n^{2H-2} \text{ as } n \to \infty.$$

The above properties, contrarily to Brownian motion, make fBm as a potential candidate to model for noise. However, since a fBm with $H \neq \frac{1}{2}$ is neither a semimartingale nor a Markov process, we cannot apply stochastic calculus developed by Itô. This is the main difficulty in studying fractional stochastic systems. The reader can consult Mishura [19] and the references therein for a more complete presentation of this subject.

The stochastic Ginzburg-Landau equations with additive colored noise and without impulses have been investigated by many authors before (see, for example, [20, 21]). We consider a new fractional stochastic version of impulsive Ginzburg-Landau equation (1.2) that reads

$$\begin{cases} dX_t = \big(a(t)X_t - b(t)X_t^3\big)dt + \sigma(t)X_t dW_t^H, \ \ t \neq t_k, \ \ k \in \mathbb{N}, \\ X_{t_k^+} - X_{t_k} = \lambda_k X_{t_k}, \ \ k \in \mathbb{N}, \end{cases} \tag{4.1}$$

where W_t^H is a fractional Brownian motion with Hurst index $H \in (\frac{1}{2}, 1)$. The fractional stochastic integral $\int\limits_0^t \sigma(s)X_s dW_s^H$ should be interpreted as a limit in $L^2(\Omega)$ of semimartingales (see, [22, Definition 2.1]).

Based on the recent advances in fractional stochastic differential equations without impulses, the existence and uniqueness of the solution of (4.1) can be shown easily in the Theorem below.

Theorem 4.1. *The unique solution of the fractional impulsive stochastic Ginzburg-Landau equation (4.1) is given by*

$$X_t = \frac{\prod\limits_{0<t_k<t} (1+\lambda_k) e^{\int\limits_0^t a(s)ds + \int\limits_0^t \sigma(s)dW_s^H}}{\left(X_0^{-2} + 2\int\limits_0^t b(s) \prod\limits_{0<t_k<s} (1+\lambda_k)^2 e^{2\int\limits_0^s a(u)du + 2\int\limits_0^t \sigma(u)dW_u^H} ds \right)^{\frac{1}{2}}}.$$

Proof. We first consider the following fractional stochastic differential equation without impulses

$$dY_t = \left(a(t)Y_t - b(t) \prod_{0<t_k<t} (1+\lambda_k)^2 Y_t^3 \right) dt + \sigma(t)Y_t dW_t^H, \ \ Y_0 = X_0. \tag{4.2}$$

It is known from [22, Theorem 3.2] that the solution of (4.2) can be explicitly found. More concretely, we have

$$Y_t = \frac{e^{\int\limits_0^t a(s)ds + \int\limits_0^t \sigma(s)dW_s^H}}{\left(X_0^{-2} + 2\int\limits_0^t b(s) \prod\limits_{0<t_k<s} (1+\lambda_k)^2 e^{2\int\limits_0^s a(u)du + 2\int\limits_0^t \sigma(u)dW_u^H} ds \right)^{\frac{1}{2}}}. \tag{4.3}$$

Consequently, $X_t = \prod\limits_{0<t_k<t} (1+\lambda_k) Y_t$ is a unique solution of (4.1). $\square$

Remark 4.1. Although the Theorem 4.1 is very similar to Theorem 2.1, the obtained results in Section 3 are not easy to extend to (4.1). The main reasons are due to the complexity of Itô formula and the lack of the exponential martingale inequalities in the context of stochastic calculus with respect to fBm.

5. Conclusion

In this paper, a stochastic Ginzburg-Landau equation with impulsive effects has been investigated. Our contributions in this paper include:

- An explicit expression for the solution which points out that the dynamic of impulsive equation is very different from one of its original equation.
- A sufficient condition under which the solution is almost surely exponentially convergent. Furthermore, Theorems 3.2 and 3.3 tell us that at infinity, the solution will not grow faster than $t^{1+\varepsilon}$ and will not decay faster than $t^{-(\frac{\bar{\sigma}^2}{2\underline{c}}+\varepsilon)}$ for any $\varepsilon > 0$.
- A first attempt to study an impulsive stochastic Ginzburg-Landau equation driven by fractional Brownian motion.

In this sense, we partly enrich the knowledge of the theory of Ginzburg-Landau equations.

Acknowledgments. The author would like to thank the anonymous referees for their valuable comments for improving the paper.

This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.02-2011.12 (16-101).

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