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A Class of Fractional Stochastic Differential Equations

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Abstract. In this paper we consider the fractional case of a class of stochastic differential equations that has many important applications. Based on an approximation approach we solve the equation with polynomial drift and fractional noise. The explicit solution is found and some applications are investigated.

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1. Introduction

The fractional Brownian motion (fBm) of Hurst parameter $H \in (0,1)$ is a centered Gaussian process $B^H = \{B_t^H, t \geq 0\}$ with the covariance function $R_H(t,s) = E[B_t^H B_s^H]$

$$R_H(t,s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

In the case where $H = \dfrac{1}{2}$, the process B^H is a standard Brownian motion. If $H \neq \dfrac{1}{2}$, B^H is neither a semimartingale nor a Markov process and the stochastic calculus developed by Itô cannot be applied. There are various approaches to fractional stochastic calculus by using some difficult tools such as: Malliavin calculus (see, for instance [1, 4]), theory of Wick product [5]. However, it is not easy to find explicit solutions from these methods for many practical problems.

In this paper using an the approximation approach in $L^2(\Omega)$ we investigate a class of fractional stochastic equations of form

$$dX_t = (aX_t^n + bX_t)dt + cX_t dB_t\,, \tag{1.1}$$

where $n \in \mathbb{N}, n \geq 2$ and B_t is a fractional Brownian motion of Liouville form that is defined below.

This equation is a generalization of many important equations such as: the Black-Sholes equation in mathematical finance, the Ginzburg-Landau equation in the theoretical physics, the Verlhust equation in population study.

In [8] Mandelbrot has given a representation of B^H of the form:

$$B_t^H = \frac{1}{\Gamma(1+\alpha)}\big[U_t + B_t\big]\,,$$

where $U_t = \int\limits_{-\infty}^{0} \big((t-s)^\alpha - (-s)^\alpha\big)dW_s,\ B_t = \int\limits_0^t (t-s)^\alpha dW_s$ and $\alpha = H - \dfrac{1}{2}$. B_t is a process possessing main properties of B_t^H such as of long memory and called a fractional Brownian motion of Liouville form [2, 6].

It is known that B_t is approximated in $L^2(\Omega)$ by stochastic processes

$$\tilde{\mathrm{B}}_t = \int\limits_0^t (t-s+\varepsilon)^\alpha dW_s\,,$$

where $\tilde{\mathrm{B}}_t$ is a semimartingale and the convergence is uniform in $t \in [0,T]$.

The paper is organized as follows: In Sec. 2 we recall some important results from the approximation approach and formulate our approximation problem. Section 3 contains main results of this paper. In Sec. 4 some applications to finance and physics are introduced.

2. An Approximation Method

Our method is based on a result on approximation of the fractional process $B_t = \int\limits_0^t (t-s)^\alpha dW_s$ by semimartingales given in [9] that we recall below:

For every $\varepsilon > 0$, as in [1] we define:

$$\tilde{\mathrm{B}}_t = \int\limits_0^t (t-s+\varepsilon)^\alpha dW_s\,,\ \ \alpha = H - \frac{1}{2} \in (-\frac{1}{2}, \frac{1}{2})\,. \tag{2.1}$$

Then we have

Theorem 2.1. I. *The process* $\{\tilde{B}_t, t \geq 0\}$ *is a semimartingale. Moreover*

$$\tilde{B}_t = \alpha I(t) + \varepsilon^\alpha W_t\,, \tag{2.2}$$

where $\varphi_\varepsilon(t) = \int\limits_0^t (t-s+\varepsilon)^{\alpha-1} dW_s$ *and* $I(t) = \int\limits_0^t \varphi_\varepsilon(s)ds\,.$

II. *The process* $\tilde{B}_t$ *converges to* B_t *in* $L^2(\Omega)$ *when* ε *tend 0. This convergence is uniform with respect to* $t \in [0,T]$ *, i.e:*

$$\lim_{\varepsilon\to 0} \sup_{0\leq t\leq T} \|\tilde{\mathrm{B}}_t - B_t\|_2 = 0 \tag{2.3}$$

Proof. Refer to [9]. ■

Corollary 2.2. *For any* $p \geq 1$, *the process* $\tilde{B}_t$ *uniformly converges in* t *to* B_t *in* $L^p(\Omega)$ *when* ε *tends 0.*

Proof. Noting that $\tilde{\mathrm{B}}_t \sim \mathcal{N}(0, \widetilde{\sigma}_t^2)$ and $B_t \sim \mathcal{N}(0, \sigma_t^2)$

$$\widetilde{\sigma}_t^2 := E|\tilde{\mathrm{B}}_t|^2 = \frac{(t+\varepsilon)^{2H} - \varepsilon^{2H}}{2H} \quad \text{and} \quad \sigma_t^2 := E|B(t)|^2 = \frac{t^{2H}}{2H}\,.$$

Since $t \in [0,T]$ and $\varepsilon \to 0^+$, it follows that both $\widetilde{\sigma}_t^2$ and σ_t^2 are bounded by some positive constant. Thus, the Gaussian processes $B_\varepsilon(t)$ and $B(t)$ have null means and finite variance. Hence they have their finite moments of any order.

If $1 \leq p \leq 2$ then by applying the Lyapunov inequality, we obtain

$$\left(E|\tilde{\mathrm{B}}_t - B_t|^p\right)^{\frac{1}{p}} \leq \left(E|\tilde{\mathrm{B}}_t - B_t|^2\right)^{\frac{1}{2}} \to 0 \text{ when } \varepsilon \to 0.$$

If $p > 2$ then it follows from Hölder inequality that

$$\begin{aligned} E|\tilde{\mathrm{B}}_t - B_t|^p &\leq \left(E|\tilde{\mathrm{B}}_t - B_t|^2\right)^{\frac{1}{2}} \left(E|\tilde{\mathrm{B}}_t - B_t|^{2p-2}\right)^{\frac{1}{2}} \\ &\leq \|\tilde{\mathrm{B}}_t - B_t\|_2 \left[2^{2p-3}\left(E|\tilde{\mathrm{B}}_t|^{2p-2} + E|B_t|^{2p-2}\right)\right]^{\frac{1}{2}} \end{aligned}$$

because $E|X+Y|^p \leq 2^{p-1}(E|Y|^p + E|Y|^p)$ for any $p \geq 1$.

Since $\tilde{\mathrm{B}}_t$ and B_t have moments of any order so there exists some constant M_p depending only p such that

$$E|\tilde{\mathrm{B}}_t - B_t|^p \leq M_p \|\tilde{\mathrm{B}}_t - B_t\|_2.$$

The proof is thus complete. ■

Next, let us consider the following fractional differential equation in a complete probability space $(\Omega, \mathcal{F}, P)$

$$\begin{cases} dX_t = \big(a\,X_t^n + b\,X_t\big)\,dt + c\,X_t\,dB_t \\ \\ X_t|_{t=0} = X_0 \end{cases} \tag{2.4}$$

or

$$\begin{cases} X_t = \int_0^t \big(a\,X_s^n + b\,X_s\big)\,ds + c\int_0^t X_s\,dB_s \\ \\ X_t|_{t=0} = X_0 \end{cases} \tag{2.5}$$

where the stochastic integral $\int_0^t X_s\,dB_s$ will be defined as the L^2-limit of $\int_0^t X_s\,d\tilde{\mathrm{B}}_s$ when $\varepsilon \to 0$, if it exists. The initial value X_0 is a measurable random variable independent of $\{B_t : 0 \le t \le T\}$.

As we said in the introduction, for the fractional stochastic calculus it is not easy to find explicit solutions of fractional stochastic differential equations. In order to avoid this difficulty and moreover, because $\tilde{\mathrm{B}}_t \rightrightarrows B_t$ it will be fully natural to approximate (2.4) by the following equation

$$\begin{cases} d\tilde{\mathrm{X}}_t = \big(a\,\tilde{\mathrm{X}}_t^n + b\,\tilde{\mathrm{X}}_t\big)\,dt + c\,\tilde{\mathrm{X}}_t\,d\tilde{\mathrm{B}}_t \\ \\ \tilde{\mathrm{X}}_t|_{t=0} = X_0 \end{cases} \tag{2.6}$$

And then the solution of equation (2.4) will be limit in $L^2(\Omega)$ of the solution of (2.6) when $\varepsilon \to 0$.

3. Main Results

The equation (2.6) is a stochastic differential equation driven by a semimartingale with a polynomial drift and a constant volatility. So the existence and uniqueness of its solution are assured. Using formula (2.2) we can rewrite equation (2.6) as follows

$$\begin{cases} d\tilde{\mathrm{X}}_t = \big(a\,\tilde{\mathrm{X}}_t^n + b\,\tilde{\mathrm{X}}_t + c\,\alpha\,\varphi^{\varepsilon}(t)\tilde{\mathrm{X}}_t\big)\,dt + c\,\varepsilon^{\alpha}\,\tilde{\mathrm{X}}_t\,dW_t \\ \\ \tilde{\mathrm{X}}_t|_{t=0} = X_0 \end{cases} \tag{3.1}$$

We have the following theorem.

Theorem 3.1 *The solution of equation (2.6) can be explicitly given by*

$$\tilde{X}_t = e^{(b-\frac{1}{2}c^2\varepsilon^{2\alpha})t + c\,\tilde{\mathrm{B}}_t}\left(X_0^{1-n} + (1-n)a\int_0^t e^{(n-1)\left((b-\frac{1}{2}c^2\varepsilon^{2\alpha})s + c\,\tilde{\mathrm{B}}_s\right)}ds\right)^{\frac{1}{1-n}}$$

Proof. Put

$$Y_t = e^{-c\,\varepsilon^\alpha W_t}$$

According to the Itô formula we have:

$$dY_t = Y_t\big(\frac{1}{2}c^2\varepsilon^{2\alpha}dt - c\,\varepsilon^\alpha dW_t\big) \tag{3.2}$$

We consider $Z_t = \tilde{\mathrm{X}}_t Y_t$ and then an application of the integration-by-part formula gives us

$$\begin{aligned} dZ_t &= \tilde{\mathrm{X}}_t dY_t + Y_t d\tilde{\mathrm{X}}_t - c^2\varepsilon^{2\alpha}\tilde{\mathrm{X}}_t Y_t\,dt \\ &= \Big\{a\,e^{(n-1)c\,\varepsilon^\alpha W_t}(Z_t)^n + \big(b + c\,\alpha\,\varphi^\varepsilon(t) - \frac{1}{2}\,c^2\,\varepsilon^{2\alpha}\big)Z_t\Big\}dt \end{aligned} \tag{3.3}$$

This is an ordinary Bernouilli equation of the form

$$Z'_t = P(t)Z_t^n + Q(t)Z_t$$

and the solution Z_t is given by

$$Z_t = e^{\int\limits_0^t Q(u)du}\Bigg(Z_0 + \int\limits_0^t (1-n)P(s)e^{(n-1)\int\limits_0^s Q(u)du}ds\Bigg)^{\frac{1}{1-n}}$$

where $P(t) = a\,e^{(n-1)\,c\,\varepsilon^\alpha\,W_t}$, $Q(t) = b - \frac{1}{2}\,c^2\,\varepsilon^{2\alpha} + c\,\alpha\,\varphi^\varepsilon(t)$

and $\int\limits_0^t Q(u)du = (b - \frac{1}{2}c^2\varepsilon^{2\alpha})t + c\,\alpha\,I(t)$.

Hence, the solution Z_t of equation (3.3) can be expressed as

$$Z_t = e^{(b-\frac{1}{2}c^2\varepsilon^{2\alpha})t+c\,\alpha\,I(t)}\Bigg(Z_0+(1-n)a\int\limits_0^t e^{(n-1)\big((b-\frac{1}{2}c^2\varepsilon^{2\alpha})s+c\,\alpha\,I(s)+c\varepsilon^\alpha W(s)\big)}ds\Bigg)^{\frac{1}{1-n}}$$

Combining the latest expression and $\tilde{\mathrm{B}}_t = \alpha I(t) + \varepsilon^\alpha W_t$ we obtain the solution of the approximation equation (2.6)

$$\tilde{\mathrm{X}}_t = e^{(b-\frac{1}{2}c^2\varepsilon^{2\alpha})t+c\,\tilde{\mathrm{B}}_t}\Bigg(X_0^{1-n} + (1-n)a\int\limits_0^t e^{(n-1)\big((b-\frac{1}{2}c^2\varepsilon^{2\alpha})s+c\,\tilde{\mathrm{B}}_s\big)}ds\Bigg)^{\frac{1}{1-n}}$$

The proof is thus complete. ■

Theorem 3.2. *Suppose that X_0 is a random variable such that $X_0 > 0$ a.s and $E[X_0^{2n}] < \infty$. If $H > \dfrac{1}{2}$ and $a \leq 0$ then the stochastic process X_t^* defined by*

$$X_t^* = e^{bt+c\,B_t}\left(X_0^{1-n} + (1-n)a\int_0^t e^{(n-1)\,(bs+c\,B_s)}ds\right)^{\frac{1}{1-n}} \tag{3.4}$$

is the limit in $L^2(\Omega)$ of $\tilde{X}_t$. This limit is uniform with respect to $t \in [0,T]$.

Proof. Put $\theta_\varepsilon(t) = e^{(b-\frac{1}{2}c^2\varepsilon^{2\alpha})t+c\,\tilde{\mathrm{B}}_t}$ and $\theta(t) = e^{bt+c\,B_t}$ then it is clear that for each $m \geq 1$ there exists a finite constant $M_m > 0$ such that $E[\theta_\varepsilon^m(t)] \leq M_m$, $E[\theta^m(t)] \leq M_m$ for every $t \in [0,T]$. Indeed,

$$E[\theta^m(t)] = e^{mbt}E[e^{m\,c\,B_t}] = e^{mbt}e^{\frac{1}{2}(m\,c\,\mathrm{b}_t)^2} = e^{mbt+\frac{1}{2}m^2c^2\mathrm{b}_t^2} < \infty,$$

and

$$E[\theta_\varepsilon^m(t)] = e^{m(b-\frac{1}{2}c^2\varepsilon^{2\alpha})t+\frac{1}{2}m^2c^2\tilde{\mathrm{b}}_t^2} < \infty\,.$$

Moreover, applying the Hölder inequality we have following estimates for any $m,k \geq 1$:

$$E[\theta_\varepsilon^m(t)\,\theta^k(t)] \leq (E|\theta_\varepsilon(t)|^{2m})^{\frac{1}{2}}\,(E|\theta(t)|^{2k})^{\frac{1}{2}}$$

So there exists a finite constant $M_{m,k} > 0$ such that

$$E[\theta_\varepsilon^m(t)\,\theta^k(t)] \leq M_{m,k}\ \forall\ t \in [0,T]\,. \tag{3.5}$$

We now can prove that $\theta_\varepsilon(t) \overset{L^2}{\rightarrow} \theta(t)$ uniformly with respect to $t \in [0,T]$, i.e:

$$\lim_{\varepsilon\to 0}\sup_{0\leq t\leq T}\|\theta_\varepsilon(t) - \theta(t)\|_2 = 0\,. \tag{3.6}$$

Indeed, we see that

$$\begin{aligned}\|\theta_\varepsilon(t) - \theta(t)\|_2 &\leq \|\theta(t)\|_4\|\exp\big(-\frac{1}{2}c^2\varepsilon^{2\alpha}t + c(\tilde{\mathrm{B}}_t - \mathrm{B}_t)\big) - 1\|_4\\ &\leq M_4\|\exp\big(-\frac{1}{2}c^2\varepsilon^{2\alpha}t + c(\tilde{\mathrm{B}}_t - \mathrm{B}_t)\big) - 1\|_4\end{aligned} \tag{3.7}$$

Using the relation $e^x - 1 = x + o(x)$, we obtain

$$\begin{aligned}&\|\exp\big(-\frac{1}{2}c^2\varepsilon^{2\alpha}t + c(\tilde{\mathrm{B}}_t - \mathrm{B}_t)\big) - 1\|_4\\ &\leq \| -\frac{1}{2}c^2\varepsilon^{2\alpha}t + c(\tilde{\mathrm{B}}_t - \mathrm{B}_t)\|_4 + \|o(...)\|_4\\ &\leq \frac{1}{2}c^2\varepsilon^{2\alpha}T + \|c(\tilde{\mathrm{B}}_t - \mathrm{B}_t)\|_4 + \|o(...)\|_4\end{aligned} \tag{3.8}$$

and thus (3.6) follows from Corollary 2.2

We have also that $\int\limits_0^t \theta_\varepsilon^{n-1}(s)ds \xrightarrow{L^2} \int\limits_0^t \theta^{n-1}(s)ds$ uniformly with respect to $t \in [0,T]$. Indeed, we have the following estimate:

$$\begin{aligned}
&E\left|\int\limits_0^t \theta_\varepsilon^{n-1}(s)ds - \int\limits_0^t \theta^{n-1}(s)ds\right|^2 \\
&\leq t\int\limits_0^t E|\theta_\varepsilon^{n-1}(t) - \theta^{n-1}(t)|^2 ds \;\forall\, t \in [0,T]\,.
\end{aligned} \tag{3.9}$$

Once again, an application of the Hölder inequality yields for every $t \in [0,T]$

$$\begin{aligned}
&E|\theta_\varepsilon^{n-1}(t) - \theta^{n-1}(t)|^2 \\
&= E[|\theta_\varepsilon(t) - \theta(t)|\, A_\varepsilon(t)] \\
&\leq \|\theta_\varepsilon(t) - \theta(t)\|_2\, \|A_\varepsilon(t)\|_2
\end{aligned} \tag{3.10}$$

where $A_\varepsilon(t) = |\theta_\varepsilon(t) - \theta(t)|\big(\theta_\varepsilon^{n-2}(t) + \theta_\varepsilon^{n-3}(t)\,\theta(t) + \ldots + \theta^{n-2}(t)\big)^2$.

Using inequalities of the form (3.5) we see that there exists a finite constant $M_n > 0$ such that

$$\|A_\varepsilon(t)\|_2 \leq M_n \;\forall\, t \in [0,T]\,. \tag{3.11}$$

It follows from (3.9),(3.10) and (3.11) that

$$\begin{aligned}
&E\left|\int\limits_0^t \theta_\varepsilon^{n-1}(s)ds - \int\limits_0^t \theta^{n-1}(s)ds\right|^2 \\
&\leq M_n\, t^2 \sup_{0\leq t\leq T} \|\theta_\varepsilon(t) - \theta(t)\|_2 \\
&\leq M_n\, T^2 \sup_{0\leq t\leq T} \|\theta_\varepsilon(t) - \theta(t)\|_2 \;\;\forall\, t \in [0,T]\,.
\end{aligned} \tag{3.12}$$

The late inequality assures that

$$\sup_{0\leq t\leq T}\left\|\int\limits_0^t \theta_\varepsilon^{n-1}(s)ds - \int\limits_0^t \theta^{n-1}(s)ds\right\|_2 \to 0 \;\text{ as }\; \varepsilon \to 0\,.$$

Put $\eta_\varepsilon(t) = X_0^{1-n} + a(1-n)\int\limits_0^t \theta_\varepsilon^{n-1}(s)ds$ and $\eta(t) = X_0^{1-n} + a(1-n)\int\limits_0^t \theta^{n-1}(s)ds$. From results above we can see that $\eta_\varepsilon(t) \xrightarrow{L^2} \eta(t)$ uniformly with respect to $t \in [0,T]$. Next we will show that

$$\eta_\varepsilon^{\frac{1}{1-n}}(t) \xrightarrow{L^2} \eta^{\frac{1}{1-n}}(t) \text{ uniformly with respect to } t \in [0,T]\,. \tag{3.13}$$

Indeed, since $a \leq 0$, we have $\eta_\varepsilon(t) \geq X_0^{1-n}$ and $\eta(t) \geq X_0^{1-n}$ a.s for every $t \in [0,T]$.

The theorem of finite increments applied to the function $g(x) = x^{\frac{1}{1-n}}$ yields

$$\left\| \eta_\varepsilon^{\frac{1}{1-n}}(t) - \eta^{\frac{1}{1-n}}(t) \right\|_2 \leq \frac{1}{n-1} \left\| X_0^n (\eta_\varepsilon(t) - \eta(t)) \right\|_2 .$$

By an argument analogous to the previous one, we get

$$\left\| \eta_\varepsilon^{\frac{1}{1-n}}(t) - \eta^{\frac{1}{1-n}}(t) \right\|_2 \leq M \; \|\eta_\varepsilon(t) - \eta(t)\|_2 \;\; \forall \; t \in [0,T] .$$

where $M > 0$ is a finite constant. And (3.13) follows from this estimate.

As a consequence we have following assertion

$$\tilde{\mathrm{X}}_t = \theta_\varepsilon(t)\eta_\varepsilon^{\frac{1}{1-n}}(t) \xrightarrow{L^2} \theta(t)\eta^{\frac{1}{1-n}}(t) = X^*(t) .$$

The proof of theorem is thus complete. ∎

4. Applications

We can easily verify that some famous equations can be considered as particular cases of our fractional equation studied in this paper.

1. The fractional Black-Scholes model given by

$$dX_t = \mu X_t \, dt + \nu X_t \, dB_t ,$$

$(a = 0, b = \mu, c = \nu)$

$$X_t = X_0 \, e^{\nu B_t + \mu \, t} . \tag{4.1}$$

2. The fractional Verhulst equation

$$dX_t = (-X_t^2 + \lambda X_t) \, dt + \sigma X_t \, dB_t ,$$

$(a = -1, b = \lambda, c = \sigma, n = 2)$

$$X_t = e^{\sigma B_t + \lambda t} \big(X_0^{-1} + (-1)(-1) \int\limits_0^t e^{\sigma B_s + \lambda s} ds \big)^{-1}$$

$$X_t = \frac{X_0 \, e^{\sigma B_t + \lambda t}}{1 + X_0 \int\limits_0^t e^{\sigma B_s + \lambda s} ds}$$

3. The fractional Ginzburg-Landau equation

$$dX_t = (-X_t^3 + (\alpha + \frac{\sigma^2}{2})X_t)\,dt + \sigma X_t\,dB_t\,,$$

$(a = -1, b = \alpha + \frac{\sigma^2}{2}, c = \sigma, n = 3\,)$

$$X_t = \frac{X_0\,e^{\sigma B_t + (\alpha + \frac{\sigma^2}{2})t}}{\left(1 + 2X_0^2 \int\limits_0^t e^{\sigma B_s + (\alpha + \frac{\sigma^2}{2})s}\right)^{\frac{1}{2}}}$$

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