Asymptotic behavior of linear fractional stochastic differential equations with time-varying delays

Nguyen Tien Dung

Department of Mathematics, FPT University
8 Ton That Thuyet, Cau Giay, Hanoi, VietNam.
Email: dung_nguyentien10@yahoo.com Cc: dungnt@fpt.edu.vn

Abstract

In this paper we give a sufficient condition for the exponential asymptotic behavior of solutions of a general class of linear fractional stochastic differential equations with time-varying delays. Our obtained results allow us to employ the theories developed for the deterministic systems and to illustrate this, some examples are provided.
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1. Introduction

It is known that stochastic differential equations (SDEs) play a very important role in formulation and analysis of many phenomena in economic and finance, in physics, mechanics, electric and control engineering, etc. Also, SDEs with delays are frequently used to model the objects where the future state depends not only on the present state but also on its past states. As is well known, the explicit solutions of SDEs with or without delays can rarely be obtained. Therefore, it is necessary to investigate numerical approximations and qualitative properties of the solution such as the asymptotic behavior of the solutions, the existence of positive solutions and periodic solutions, etc.

The solution of Itô SDEs is a semimartingale as well a Markov process. However, many objects in real world are not always such processes since they have long-range aftereffects. Since the work of Mandelbrot and Van Ness [18] there has been an increased interest in stochastic models based on the fractional Brownian motion than Brownian motion. A fractional Brownian motion (fBm) of Hurst parameter $H \in (0,1)$ is a centered Gaussian process $B^H = \{B^H(t), t \geq 0\}$ with the covariance function $R_H(t,s) = E[B^H(t)B^H(s)]$

$$R_H(t,s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}).$$

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It is known that, contrarily to Brownian motion, fBm with Hurst index $H \neq \frac{1}{2}$ is a Gaussian process that has a memory. More precisely, let $\rho_H(n) := E(W_1^H(W_{n+1}^H - W_n^H))$, then

$$\rho_H(n) \approx H(2H-1)n^{2H-2} \text{ as } n \to \infty.$$

Thus, according to Beran's definition [1], if $H > \frac{1}{2}$ then fBm is called a long-memory process and presents an aggregation behavior. If $H < \frac{1}{2}$ then fBm is called a short-memory process. The long-memory property make fBm as a potential candidate to model for noise in mathematical finance (see [7]); in biology (see [4, 22]); in geophysics (see [23]); in hydrology (see [16]); in communication networks (see, for instance [26]); the analysis of global temperature anomaly [24]; electricity markets [25], etc. We also refer to [13] for a short survey on a series of studies that have been made to detect fractal long-memory in finance.

It is known that fBm is a generalization of Brownian motion, it reduces to Brownian motion when $H = \frac{1}{2}$. In fact, the stability theory of SDEs with delays is now well established, but for SDEs driven by fBm with delays has not yet been developed. As stated in [5], there exist only a few papers published in this field and the most of the papers is to prove the existence and uniqueness of the solution (for instance, see [12, 13, 14, 17, 20]).

In [18] Mandelbrot et al. has given a representation of $B^H(t)$ of the form:

$$B^H(t) = \frac{1}{\Gamma(1+\alpha)}\left(U(t) + \int_0^t (t-s)^\alpha dW(s)\right),$$

where $\alpha = H - \frac{1}{2}, U(t)$ is a stochastic process of absolutely continuous trajectories, and $W^H(t) := \int_0^t (t-s)^\alpha dW(s)$ is called a fBm of the Liouville form (LfBm). Because a LfBm shares many properties of a fBm (except that it has non-stationary increments) and for simplicity we use $W^H(t)$ standing for $B^H(t)$ throughout this paper.

The aim of this paper is to study asymptotic behavior of the solution of the following scalar equation

$$dX(t) = \sum_{k=1}^{N} a_k(t)X(r_k(t))dt + \sigma(t)dW^H(t) \ , t > 0 \tag{1.1}$$

with the initial function and the initial value

$$X(t) = \phi(t) \ , t < 0 \ , \ \ X(0) = X_0 = \text{constant}, \tag{1.2}$$

where $W^H(t)$ is a LfBm with $H > \frac{1}{2}$. The delays $r_k(t), k = 1, 2..., N$ are measurable deterministic functions satisfying usual assumptions: $r_k(t) \leq t, \sup_{t\geq 0}[t - r_k(t)] < \bar{r} < \infty$ and $\phi : (-\infty, 0) \to \mathbf{R}$ is a measurable bounded deterministic function. Since our investigation is focused on stability problems, we will assume that the coefficients $a_k(t), k = 1, 2, ..., N$ satisfy the conditions that assure the existence and uniqueness of the solution of the equation (1.1) with the initial conditions (1.2). For example, a simple condition is essentially boundedness of coefficients on $[0, +\infty)$.

This paper is organized as follows: In Section 2, we recall the definition of a stochastic integral with respect to LfBm from an approximate approach. Section 3 contains the main results of this paper: we prove a representation formula for the solution and then based on the obtained formula, a sufficient condition is derived to ensure the p-th mean exponentially boundedness and the almost surely exponential convergence of the solution to zero. The conclusion and some examples are given in Section 4.

2. Preliminaries

In the last few decades, many different ways have been introduced to construct the fractional stochastic calculus (see, for instance, [8]). The main difficulties in studying fractional stochastic systems are that we cannot apply stochastic calculus developed by Itô since fBm is neither a Markov process nor a semimartingale, except for $H = \frac{1}{2}$. Recently, an approximate approach has been developed to avoid those difficulties (see, [10, 11, 12] and the references therein). Let us recall some fundamental results about this approach.

For every $\varepsilon > 0$ we define

$$W^{H,\varepsilon}(t) = \int_0^t (t - s + \varepsilon)^{\alpha} dW(s). \tag{2.1}$$

In [10], author proved that $W^{H,\varepsilon}(t)$ is a semimartingale with the following decomposition

$$W^{H,\varepsilon}(t) = \varepsilon^{\alpha} W(t) + \int_0^t \varphi^{\varepsilon}(s) ds, \tag{2.2}$$

where $\varphi^{\varepsilon}(s) = \int_0^s \alpha(s + \varepsilon - u)^{\alpha-1} dW(u)$. Moreover, $W^{H,\varepsilon}(t)$ converges to $W^H(t)$ in $L^p(\Omega)$, $p > 1$ uniformly in $t \in [0, T]$ as $\varepsilon \to 0$:

$$E|W^{H,\varepsilon}(t) - W^H(t)|^p \leq c_{p,T} \varepsilon^{pH}.$$

For f is a deterministic function in $L^2[0,T]$, from the decomposition (2.2) we have

$$\int_0^t f(s)dW^{H,\varepsilon}(s) = \int_0^t \varepsilon^\alpha f(s)dW(s) + \int_0^t \int_0^s \alpha f(s)(s+\varepsilon-u)^{\alpha-1}dW(u)ds$$
$$= \int_0^t \varepsilon^\alpha f(s)dW(s) + \int_0^t \int_u^t \alpha f(u)(u+\varepsilon-s)^{\alpha-1}dudW(s) \quad (2.3)$$

As $\varepsilon \to 0$, each term in the right-hand side of (2.3) converges in $L^2(\Omega)$ to the same term where $\varepsilon = 0$. Then, it is "natural" to define (we refer the reader to [6, 11] for a general definition)

Definition 2.1. For f is a deterministic function in $L^2[0,T]$. The stochastic integral of f with respect to LfBm is defined by

$$\int_0^t f(s)dW^H(s) := \lim_{\varepsilon\to 0}\int_0^t f(s)dW^{H,\varepsilon}(s) = \alpha\int_0^t \int_s^t f(u)(u-s)^{\alpha-1}dudW(s). \quad (2.4)$$

Remark 2.1. By applying Hölder inequality we have the following estimate

$$E\Big(\int_0^t f(s)dW^H(s)\Big)^2 = \alpha^2\int_0^t \Big(\int_s^t f(u)(u-s)^{\alpha-1}du\Big)^2 ds$$
$$\leq \alpha^2\int_0^t \Big(\int_s^t f(u)^2(u-s)^{\alpha-1}du\Big)\Big(\int_s^t (u-s)^{\alpha-1}du\Big)ds$$
$$\leq t^\alpha\int_0^t f(u)^2u^\alpha du \leq t^{2\alpha}\int_0^t f(u)^2du. \quad (2.5)$$

3. The main result

Since LfBm is non-semimartingale, we cannot directly apply traditional methods (for instance, method of Lyapunov functionals) to study the stability of the solution of (1.1). We consider the deterministic differential equations with time-varying delays:

$$dx(t) = \sum_{k=1}^N a_k(t)x(r_k(t))dt. \quad (3.1)$$

Following the Mao' idea in [21], we assume this delay equation is exponentially stable. And then by semimartingale approximate approach we can prove that the variation of constants formula is still valid for (1.1). Naturally, desired results of this paper can be found via the well-known results of deterministic systems (3.1). Let us recall some fundamental concepts.

Definition 3.1. I. A solution $Z(t,s)$ of the equation

$$\begin{cases} dZ(t) = \sum\limits_{k=1}^{N} a_k(t) Z(r_k(t))dt \ , \ t \geq s \\ Z(t) = 0 \ , \ t < s \ , \quad Z(s) = 1 \end{cases} \tag{3.2}$$

is called the fundamental solution of (3.1).

II. The equation (3.1) is exponentially stable, if there exist $K > 0, \lambda > 0$ such that the fundamental solution $Z(t,s)$ has the estimate

$$|Z(t,s)| \leq K e^{-\lambda(t-s)} \ , \quad t \geq s \geq 0$$

The following theorem plays a key role in this paper.

Theorem 3.1. *Let $Z(t,s)$ be the solution of (3.2). Then the solution of (1.1) is a Gaussian process and admits the following representation*

$$X(t) = Z(t,0)X_0 + \int_0^t Z(t,s) \sum_{k=1}^{N} a_k(s)\phi(r_k(s))ds + \int_0^t Z(t,s)\sigma(s)dW^H(s), \ \ t > 0, \tag{3.3}$$

where $\phi(t) = 0, t \geq 0$.

Proof. Using the same arguments as in [11, 12] we can see that the solution $X(t)$ of (1.1) can be approximated in $L^2(\Omega)$ by the semimartingale $X^{\varepsilon}(t)$ which solves

$$dX^{\varepsilon}(t) = \sum_{k=1}^{N} a_k(t) X^{\varepsilon}(r_k(t))dt + \sigma(t)dW^{H,\varepsilon}(t) \ , t > 0 \tag{3.4}$$

with the initial conditions $X^{\varepsilon}(t) = \phi(t) \, , t < 0 \, , X^{\varepsilon}(0) = X_0$. From the decomposition (2.2), the equation (3.4) can be rewrote as follows:

$$dX^{\varepsilon}(t) = \left(\sum_{k=1}^{N} a_k(t) X^{\varepsilon}(r_k(t)) + \sigma(t)\varphi^{\varepsilon}(t) \right) dt + \sigma(t)\varepsilon^{\alpha} dW(t) \ , t > 0. \tag{3.5}$$

It is clear that (3.5) is a classical Itô stochastic delay differential equation, its solution admits the following representation (see, for instance [19])

$$X^{\varepsilon}(t) = Z(t,0)X_0 + \int_0^t Z(t,s)\sigma(s)\varphi^{\varepsilon}(s)ds + \int_0^t Z(t,s)\sum_{k=1}^{N} a_k(s)\phi(r_k(s))ds$$
$$+ \int_0^t Z(t,s)\sigma(s)\varepsilon^{\alpha}dW(s),\ t \geq 0,$$

or equivalently

$$X^{\varepsilon}(t) = Z(t,0)X_0 + \int_0^t Z(t,s)\sum_{k=1}^{N} a_k(s)\phi(r_k(s))ds + \int_0^t Z(t,s)\sigma(s)dW^{H,\varepsilon}(s),\ t \geq 0. \quad (3.6)$$

The formula (3.3) follows directly from (3.6) by taking the limit in $L^2(\Omega)$ as $\varepsilon \to 0$. The Gaussian property of the solution is obvious because $Z(t,s)$ and $\sigma(s)$ are deterministic. □

Lemma 3.1. *Suppose that $f \in C(\mathbb{R}^+, \mathbb{R}^+) \cap L^1(\mathbb{R}^+)$ satisfies*

$$\int_0^\infty f(t)e^{\gamma t}dt < \infty \text{ for some } \gamma > 0.$$

If $\lambda > 0$ and $\lambda' = \lambda \wedge \gamma$ then

$$\int_0^t e^{-\lambda(t-s)}f(s)ds < e^{-\lambda' t}\int_0^\infty f(s)e^{\gamma s}ds.$$

Proof. It is easy to prove by using the tools from the real functional analysis. □

Theorem 3.2. *Suppose that the equation (3.1) is exponentially stable and that the following conditions hold*

$$\int_0^\infty |\sum_{k=1}^{N} a_k(s)\phi(r_k(s))|e^{\lambda_1 s}ds < \infty \text{ for some } \lambda_1 > 0, \quad (3.7)$$

$$\int_0^\infty \sigma^2(s)e^{2\lambda_2 s}ds < \infty \text{ for some } \lambda_2 > 0. \quad (3.8)$$

Then

I. *there exists $\gamma > 0$ and $M_p(X_0) > 0$ such that for each $p > 0$, the solution of (1.1) satisfies*

$$E|X(t)|^p \leq M_p(X_0)e^{-p\gamma t}\ ,\ t \geq 0. \tag{3.9}$$

II. *In addition, we assume that*

$$\int_0^\infty \sum_{k=1}^N a_k^2(s)e^{2\beta_0 s}ds < \infty \ \ \textit{for some}\ \ \beta_0 > -1. \tag{3.10}$$

Then the solution of (1.1) is almost surely exponentially convergent, i.e. there exists $\beta_1 > 0$ such that

$$\limsup_{t\to\infty} \frac{1}{t}\log|X(t)| \leq -\beta_1\ ,\ a.s. \tag{3.11}$$

Proof. **I.** Let $Z(t,s)$ be the solution of (3.2). Since Eq. (3.1) is exponentially stable, there exist $K > 0$ and $\lambda > 0$ such that

$$|Z(t,s)| \leq Ke^{-\lambda(t-s)}\ ,\forall\ t \geq s \geq 0. \tag{3.12}$$

From (3.3) and using the estimates (2.5) and (3.12) we have

$$\begin{gathered} E|X(t) - Z(t,0)X_0 - \int_0^t Z(t,s)\sum_{k=1}^N a_k(s)\phi(r_k(s))ds|^2 \\ \leq t^{2\alpha}\int_0^t |Z(t,s)\sigma(s)|^2 ds \leq K^2t^{2\alpha}\int_0^t e^{-2\lambda(t-s)}|\sigma(s)|^2 ds. \end{gathered} \tag{3.13}$$

Let $\lambda' = \min(\lambda,\lambda_2)$ and $\lambda_3 = \frac{\lambda'}{2}$. By Lemma 3.1

$$\begin{gathered} E|X(t) - Z(t,0)X_0 - \int_0^t Z(t,s)\sum_{k=1}^N a_k(s)\phi(r_k(s))ds|^2 \\ \leq K^2t^{2\alpha}e^{-2\lambda' t}\int_0^\infty |\sigma(s)|^2e^{2\lambda_2 s}ds \leq K_2^2e^{-2\lambda_3 t}, \end{gathered} \tag{3.14}$$

where $K_2^2 = K_1\int_0^\infty |\sigma(s)|^2e^{2\lambda_2 s}ds$ and $K_1 = \sup_{t\in(0,\infty)} K^2t^{2\alpha}e^{-\lambda' t}$.

In order to establish (3.9) we need two technical results:
(i) Let Y denote a random variable following an $N(0,a^2)$ law. Then for any $p > 0$ we have

$$E|Y|^p = \frac{2^{p/2}\Gamma(\frac{p+1}{2})}{\Gamma(\frac{1}{2})}a^p.$$

(ii) $(x+y+z)^p \le c_p(x^p+y^p+z^p)\ \forall\ x,y,z>0$, where $c_p=1$ if $0<p\le 1$ and $c_p=3^{p-1}$ if $p>1$.

From (3.14) and by using Lemma 3.1 we now have

$$\begin{aligned} E|X(t)|^p &\le c_p\Big(E|X(t)-Z(t,0)X_0-\int_0^t Z(t,s)\sum_{k=1}^N a_k(s)\phi(r_k(s))ds|^p \\ &+|Z(t,0)X_0|^p+|\int_0^t Z(t,s)\sum_{k=1}^N a_k(s)\phi(r_k(s))ds|^p\Big) \\ &= c_p\Big(\frac{2^{p/2}\Gamma(\frac{p+1}{2})}{\Gamma(\frac{1}{2})}K_2^p e^{-p\lambda_3 t}+|X_0|^p K^p e^{-p\lambda t} \\ &+K^p e^{-p\lambda'' t}(\int_0^\infty|\sum_{k=1}^m a_k(s)\phi(r_k(s))|e^{\lambda_1 s}ds)^p\Big) \\ &\le M_p(X_0)e^{-p\gamma t}, \end{aligned}$$

where $\lambda''=\min(\lambda,\lambda_1)$, $\gamma=\min(\lambda'',\lambda_3)$ and $M_p(X_0)=c_p\Big(\frac{2^{p/2}\Gamma(\frac{p+1}{2})}{\Gamma(\frac{1}{2})}K_2^p+|X_0|^pK^p+(\int_0^\infty|\sum_{k=1}^m a_k(s)\phi(r_k(s))|e^{\lambda_1 s}ds)^p\Big)<\infty$.

II. It is enough to show that there exists a constant $\gamma_1>0$ such that

$$E\Big(\sup_{n-1\le t\le n}|X(t)|^2\Big)\le Me^{-\gamma_1 n}\ \ \forall\ n\ge 1. \tag{3.15}$$

Indeed, if (3.15) is true, using Chebyshev inequality, we have for any $\bar\gamma<\gamma_1$

$$P\Big(\sup_{n-1\le t\le n}|X(t)|^2>e^{-\bar\gamma n}\Big)\le e^{\bar\gamma n}E\Big(\sup_{n-1\le t<n}|X(t)|^2\Big)\le Me^{-(\gamma_1-\bar\gamma)n}$$

Since $\sum_{n=1}^\infty e^{-(\gamma_1-\bar\gamma)n}<\infty$, an application of Borel-Cantelli lemma yields there exists an $\Omega_0\subset\Omega$ with $P(\Omega_0)=1$ such that for any $\omega\in\Omega_0$ there exists an integer $n_0(\omega)$, when $n\ge n_0(\omega)$ and $n-1\le t\le n$,

$$|X(t)|^2\le e^{-\bar\gamma n}\le e^{-\bar\gamma t},$$

which implies desired (3.11) with $\beta_1=\frac{\bar\gamma}{2}$. The remainder of the proof is to check (3.15). We have by the inequality $(x+y+z)^2\le 3(x^2+y^2+z^2)$

$$\begin{aligned} E\Big(\sup_{n-1\le t\le n}|X(t)|^2\Big) &\le 3\Big(E|X_{n-1}|^2+E\Big(\int_{n-1}^n\sum_{k=1}^N|a_k(s)X(r_k(s))|ds\Big)^2 \\ &+E\Big(\sup_{n-1\le t\le n}\Big|\int_{n-1}^t\sigma(s)dW^H(s)\Big|^2\Big)\Big):=3(I_1+I_2+I_3). \end{aligned} \tag{3.16}$$

By the results from the part **I** we have

$$I_1 \leq M_2(X_0)e^{-2\gamma(n-1)}. \tag{3.17}$$

By the Burkholder-Davis-Gundy inequality, there exists $C_1 > 0$ such that

$$I_3 \leq C_1 \int_{n-1}^{n} \Big(\int_{r}^{n} \sigma(s)\alpha(s-r)^{\alpha-1} ds \Big)^2 dr$$
$$\leq C_1 \int_{n-1}^{n} |\sigma(s)|^2 ds \leq C_1 e^{-2\lambda_2(n-1)} \int_{n-1}^{\infty} |\sigma(s)|^2 e^{2\lambda_2 s} ds \leq C_2 e^{-2\lambda_2(n-1)}, \tag{3.18}$$

where $C_2 = C_1 \int_0^\infty |\sigma(s)|^2 e^{2\lambda_2 s} ds < \infty$. To estimate I_2, we note that

$$\int_{n-1}^{n} |a_k(s)X(r_k(s))|ds \leq \int_{n-1}^{n} |a_k(s)\phi(r_k(s))|ds + \int_{n-1}^{n} |a_k(s)X(r_k(s))|\mathbf{1}_{\{r_k(s)>0\}}(s)ds.$$

And then by $(x+y_1+...+y_N)^2 \leq (N+1)(x^2+y_1^2+....+y_N^2)$ and the results from the part **I** we have

$$I_2 \leq (N+1)\bigg(\Big(\int_{n-1}^{n} \sum_{k=1}^{N} |a_k(s)\phi(r_k(s))|ds\Big)^2 + \int_{n-1}^{n} \sum_{k=1}^{N} a_k^2(s) M_2(X_0) e^{-2r_k(s)} ds\bigg)$$
$$\leq (N+1)\bigg(\Big(e^{-\lambda_1(n-1)} \int_{n-1}^{n} \sum_{k=1}^{N} |a_k(s)\phi(r_k(s))|e^{\lambda_1 s} ds\Big)^2 + \int_{n-1}^{n} \sum_{k=1}^{N} a_k^2(s) M_2(X_0) e^{-2(s-\bar{r})} ds\bigg)$$
$$\leq (N+1)\bigg(\Big(e^{-\lambda_1(n-1)} \int_{0}^{\infty} \sum_{k=1}^{N} |a_k(s)\phi(r_k(s))|e^{\lambda_1 s} ds\Big)^2$$
$$+ M_2(X_0)e^{2\bar{r}} e^{-2(1+\beta_0)(n-1)} \int_{0}^{\infty} \sum_{k=1}^{N} a_k^2(s) e^{2\beta_0 s} ds\bigg) \leq C_3 e^{-2\lambda_4(n-1)}, \tag{3.19}$$

where $\lambda_4 = \min(\lambda_1, 1+\beta_0)$ and

$$C_3 = (N+1)\bigg(\Big(\int_{0}^{\infty} \sum_{k=1}^{N} |a_k(s)\phi(r_k(s))|e^{\lambda_1 s} ds\Big)^2 + M_2(X_0)e^{2\bar{r}} \int_{0}^{\infty} \sum_{k=1}^{N} a_k^2(s) e^{2\beta_0 s} ds\bigg) < \infty.$$

We now combine (3.16), (3.17), (3.18) and (3.19) to get

$$
\begin{aligned}
E\Big(\sup_{n-1\le t\le n}|X(t)|^2\Big) &\le 3\big(M_2(X_0)e^{-2\gamma(n-1)}+C_3e^{-2\lambda_4(n-1)}+C_2e^{-2\lambda_2(n-1)}\big)\\
&\le Me^{-\gamma_1 n},
\end{aligned}
$$

where $\gamma_1=\min(2\gamma,2\lambda_4,2\lambda_2)$ and $M=3(M_2(X_0)e^{2\gamma}+C_3e^{2\lambda_4}+C_2e^{2\lambda_2})$.

The Theorem thus is proved. □

Remark 3.1. In the above Theorem we assumed that the equation (3.1) is exponentially stable. For the sake of convenience, let us restate an excellent result made by Berezansky and Braverman in [2]: Suppose that

$$\liminf_{t\to\infty}\sum_{k=1}^{N}a_k(t)>0$$

and there exists $r_0(t)\le t$, such that for sufficiently large t

$$\int_{r_0(t)}^{t}\sum_{k=1}^{N}a_k(s)ds\le\frac{1}{e}.$$

If

$$\limsup_{t\to\infty}\sum_{k=1}^{N}\Big|\sum_{i=k}^{N}a_i(t)\Big|\Big|\int_{r_{k-1}(t)}^{r_k(t)}\sum_{i=1}^{N}|a_i(s)|ds\Big|\Big/\sum_{k=1}^{N}a_k(t)<1$$

then the equation (3.1) is exponentially stable.

4. Conclusion, examples and open remarks

In this paper, we showed that the variation of constants formula is still true for the fractional stochastic differential equations with time delays. This then allows us to give a sufficient condition for the exponential asymptotic behavior of the solution. The advantages of our method are that we can employ nice results from deterministic theory. To illustrate this, let us discuss the following two examples.

Example 1. We first consider the delayed fractional Langevin equation in physics:

$$dX(t)=\big(aX(t)+bX(t-r)\big)dt+\sigma(t)dW^H(t) \tag{4.1}$$

From [15], the necessary and sufficient conditions for exponential stability of deterministic equation $dx(t)=\big(ax(t)+bx(t-r)\big)dt$ are that every root λ of characteristic equation $\lambda-a-$

$be^{-\lambda r} = 0$ has the property that $Re\lambda < 0$. Thus, this condition, together with (3.8), implies the solution of (4.1) is almost surely exponentially convergent.

Example 2. Similarly, the solution of the equation:

$$dX(t) = -X(t) - b(1 + \cos t)X(t - r(t)) + \sigma(t)dW^H(t) \tag{4.2}$$

is almost surely exponentially convergent if $b < 2/(2 + \sqrt{2})$ (see, [3]) and the condition (3.8) holds.

Open remarks. As we said in the Introduction, the theory of SDEs driven by fBm with delays is in first stage of study and the most of the papers is devoted to the existence and uniqueness of the solution. To find deeper properties of the solution, there will a lot of works that need to do. Let us consider a simple linear equation:

$$dX(t) = \big(a(t)X(t) + b(t)X(t - r)\big)dt + \sigma(t)X(t)dW^H(t). \tag{4.3}$$

The equation (4.3) is well known as the Black-Scholes equation with or without delay in finance. The existence of its solution can be proved similarly as in [12, 13].

When $b(t) = 0$, the explicit solution of (4.3) can be found. By employing this feature, Duncan et al. [9] established sufficient conditions for asymptotic behavior of the solution. Clearly, their method can not be applied to (4.3) with $b(t) \neq 0$. On the other hand, the method used in the present paper is also can not effectively applied to (4.3). The main reason is due to the appearance of $X(t)$ in random component of the equation which leads the failure of the estimate (2.5) and the lack of Gaussian property of the solution. We refer the reader [11, Definition 2.1] to see the complexity of fractional stochastic integrations.

We conclude this paper with an open question: Whether or not a method can be used to find the sufficient conditions for asymptotic behavior of the solution of (4.3), or, more generally, for the equation

$$dX(t) = a\big(t, X(t), X(t - r)\big)dt + \sigma(t)X(t)dW^H(t).$$

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